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Modern Geometry on Artful Display

Geometry Revealed: A Jacob's Ladder to Modern Higher Geometry. *By Marcel Berger, Springer, New York, 2010, 860 pages, $89.95.*

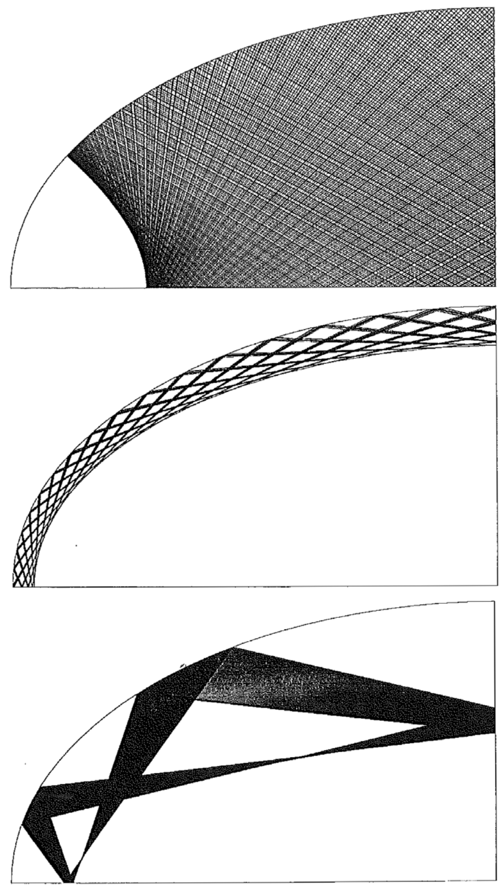
BOOK REVIEW

By Ernest Davis

This is a wonderful book, in a literal sense. Its 800 pages are full of wonders: beautiful theorems, elegant proofs, and profound theories, placed in an intricate framework of thematic and historical connections. It is like an exhibit at an art museum—room after room is filled with treasures, each worth a king's ransom, each displayed in a way that maximizes its aesthetic impact; an erudite curator has structured and annotated the whole show so as to reveal the significance and evolution of the art.

The author's central objective is to discuss geometric problems that are easily stated and visualized, but whose solution requires mathematical tools that were created for a different purpose and that lie at higher levels of abstraction. These increasing levels of abstraction are the "Jacob's ladder" of the subtitle. Each chapter takes as a point of departure a collection of basic problems in two- and three-dimensional Euclidean geometry.

How can K points be distributed on the surface of a sphere so as to maximize the distance between the closest pair? The solution is known for $K \leq 12$ and $K = 24$. What is the trajectory of a ball caroming around a billiard table? It may be periodic or ergodic; it may fill the interior of the table, or only part of the interior; it may fill the space quickly or very slowly: For the three-dimensional problem, almost nothing is known. Given the combinatorial structure of a polytope, can it be instantiated by vertices with integer coordinates and, if so, how large do the integers have to be? In two dimensions, yes, with integers of size $O(n^{3/2})$; in three dimensions, again yes, but the upper bound is very badly known; in four dimensions, not always, and if it can, the integers may have to be doubly exponential. If you follow a geodesic from a starting point on a smooth surface, how far can you go before the geodesic ceases to be the unique shortest path from the starting point (the "cut point" of the geodesic)? The complete answer is known only for a few special surfaces, such as the sphere and the torus; it is not known for the ellipsoid. Berger explores the deep mathematical issues that such problems raise; he also tracks the history of these problems—conjectures, partial solutions, errors—in remarkable detail, and provides extensive chapter bibliographies ranging from Greek mathematics to current research.



Possible trajectories for a billiard ball in an elliptical boundary. From Geometry Revealed.

The book is deliberately modeled, in part, on Hilbert and Cohn-Vossen's classic *Geometry and the Imagination*, but at a degree of mathematical depth and sophistication suitable for an expert. It assumes as primary background the author's two-volume textbook *Geometry I* and *II*, but additionally calls as needed on a broad range of mathematical domain and techniques. Proofs are given in sketches and hints; the reader is invited to fill these in. In accordance with the central theme, Berger is particularly fond of proofs in which a clever transformation or an insightful abstraction renders the statement obvious; a solution that does not "go a step up Jacob's ladder" is a disappointment. He loves to find the proper theoretical habitat for a problem; in his discussion of conics, for instance, he moves gradually upward from the geometry of the simple Euclidean plane, ending at the projective, complex plane, the setting in which theorems about conics take their purest and most general form. He is also generally punctilious about proving theorems under the weakest possible assumptions—for instance, avoiding a metric proof for a theorem in affine geometry, or not assuming third-order smoothness when second-order will suffice.

Even with 800 pages and 12 chapters, the book cannot cover all topics of modern geometry. There is nothing here on axiomatics or logical issues; little on combinatorial problems; nothing on algorithms; little on fractals; little on topology; nothing on knot theory. Riemannian and hyperbolic geometries and higher-dimensional Euclidean geometries are discussed at some length when relevant, but infinite-dimensional spaces and finite geometries are hardly mentioned. But the breadth of what is here is amazing: points and lines, circles and spheres; conics and quadrics; plane curves and smooth surfaces; convex sets; polyhedra; lattices, tilings, and packing; and the dynamics of billiards and of geodesic flow.

There are entertaining and enlightening digressions of all kinds. We learn how to manufacture a precise straight-edge and a precise spherical surface; we learn that "before 1950, the pitiable students of the terminal class of the French lycées had to know how to prove [theorems about conics] 'by hand with pure geometry' "; we learn about soap bubbles and pebbles and ornamentation on medieval columns.

The book is beautifully produced and richly illustrated, with charming hand-drawn sketches, diagrams, computer graphics, and photographs. Kudos also to Lester Senechal, the translator. The book is not necessarily intended to be read carefully cover to cover; the author recommends that you look at those chapters and topics that are particularly appealing to you. Dependencies between chapters are in general slight, so you can pick and choose. But it is worthwhile to at least skim the parts that you do not plan to study; there are treasures to be found in unexpected corners. For me, the chapters on convexity and on billiards were among the more difficult and less captivating, but in each I found a reference to a theorem that I had been seeking for twenty years.

Many people have decried the vanishing of concrete, visualizable geometry in mathematical research and education, and its replacement by abstractions. In education this certainly remains true—university courses on the kind of material in this book are rare—but the book shows that current research in geometry is active, profound, and exciting. I am only an amateur mathematician myself, and many parts of this book went over my head. I am now eager to fill in the gaps of my knowledge so that I can read the rest of it. I have learned much from reading *Geometry Revealed*; I hope in future rereadings to learn even more.

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