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Metric Geometry in Action

By Ron Kimmel

Shape matching and comparison are important for understanding the world we live in. The problem is especially challenging for non-rigid objects. Understanding the geometric nature of surfaces plays a major role in the work of a radiologist who needs to decipher an MRI image of the brain, or of a dress designer who is customizing a wedding dress. This year, geometry-capturing devices have become a popular commodity product: The Kinect is a single-pattern triangulation sensor that Microsoft has been selling with great success, and Lenovo plans on using a somewhat cruder technology based on time of flight to interact with its gaming boxes in China; Sony has joined the party with the planned release of a geometric sensor by next year, and InVision has developed a low-cost accurate multi-pattern triangulation sensor.

All this is just the tip of the iceberg. Man–machine interfaces are experiencing a revolution. As we work to acquire the geometric structure of objects, questions arise: How can we measure the similarity between two instances of the same heart at different times of a cardiac cycle? How can we match a glove and a hand, or recognize a face by its geometric structure?

As sensors improve, accurate analysis tools need to be developed. To handle such data, we try to apply tools of metric geometry analysis and turn them into computational methods. An example is the Gromov–Hausdorff distance, which quantifies the discrepancy between metric spaces. Until recently, it was applied mainly in theoretical explorations of metric geometry—in the celebrated Hamilton–Perelman proof of the Poincaré conjecture, for example.

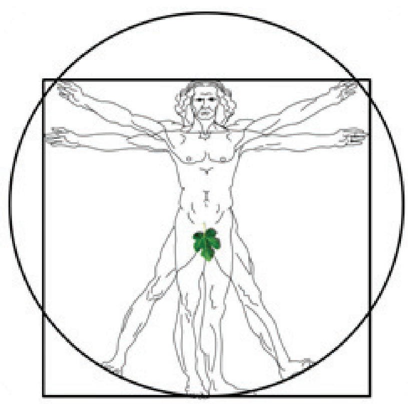

Figure 1. *Which metric best captures the invariance properties of non-rigid articulated shapes?*

An intuitive application of non-rigid ob-ject comparison is the identification of people by bio-geometric measures, or biometry. Our faces, hands, and legs change slowly in time yet still serve as an excellent identity measure when we want to distinguish between people.

Biometry appears in biblical stories and fairy tales. Jacob got the blessing of Isaac by pretending to be his brother. Little Red Riding Hood used feature-based face recognition to reach the amazing conclusion that she was talking to a wolf rather than her grandma. The prince found his true love by matching the non-rigid shape of Cinderella's foot to a rigid template—a glass slipper. Biometry obviously plays a role in modern times. O.J. Simpson was acquitted of murdering his wife because the glove found at the scene of the crime did not match his hand; recently, he allegedly confessed to committing the crime. The case is a good demonstration that shape matching must be applied with great care. So how should we match a hand with a glove?

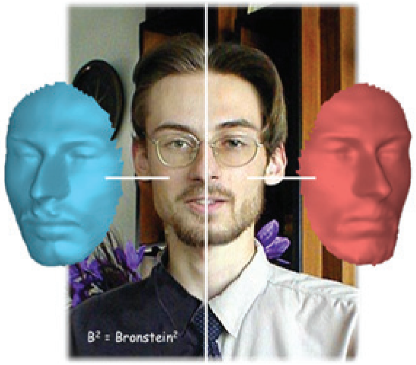

Figure 2. *Identical twins: Alex Bronstein, now a professor at Tel Aviv University (left), and Michael Bronstein, a professor at USI Lugano, in Switzerland (right).*

I should confess to a personal interest in biometry. During my first years as a professor at the Technion, I met a pair of identical twins who eventually became my students. Their project was to construct an expression-insensitive system that could distinguish between them based on the geometry of their faces. This project was an important milestone in our fascinating journey of exploring invariant measures and computable geometric structures.

Certain properties could allow us to translate the relevant shape-matching problems into computable measures. It appears that intrinsic geometry is an approximate invariant of many natural objects—a hand in different positions, for example, or a face with various expressions. We can therefore state that the intrinsic geometry of a facial surface corresponds to the identity of the person, while the extrinsic geometry reflects facial expressions. In other words, when we consider the face of a single person under various expressions, geodesic distances between surface points are better preserved than Euclidean distances.

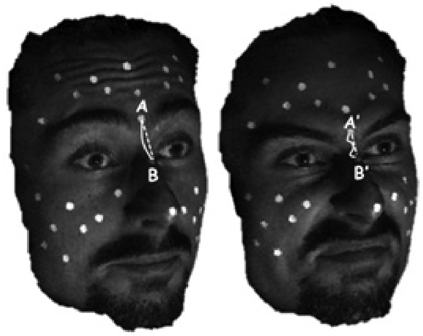

Figure 3. *Geodesic distances are less affected by expressions than Euclidean ones.*

Efficient tools for matching and comparing rigid objects, such as the iterative closest point, have existed for more than three decades. Only recently, though, have scientists considered the application of these methods to non-rigid objects. How can we extend these tools and compare the intrinsic geometry of faces?

Consider the following formulation of the Gromov–Hausdorff distance,

$$d_{\mathrm{GH}}(S,Q) = \frac{1}{2} \min_{c} \max_{\substack{(s_i,q_i)\in C \\ (s_j,q_j)\in C}} \left| d_s(s_i,s_j) - d_Q(q_i,q_j) \right|,$$

where

$$\forall s_i \ \exists q_i \text{ s.t.} (s_i,q_i) \in C$$

and

$$\forall q_i \ \exists s_i \text{ s.t.} (s_i,q_i) \in C.$$

Points from surface Q are associated to points of S (through C) such that the distances between corresponding pairs on the two surfaces are as close as possible in a min–max sense. At first glance the measure reads like a hard-to-compute permutation problem. We can explore an alternative formulation,

$$d_{\mathrm{GH}}(S,Q) = \frac{1}{2} \inf_{\substack{Z \\ \rho:S\to Z \\ \sigma:Q\to Z}} d_{\mathbf{H}}^{Z}(\rho(S),\sigma(Q)),$$

where

$$d_{\mathbf{H}}^{Z}(S,Q) \equiv \max\left\{ \sup_{s\in S} d_Z(s,Q), \sup_{q\in Q} d_Z(q,S) \right\}.$$

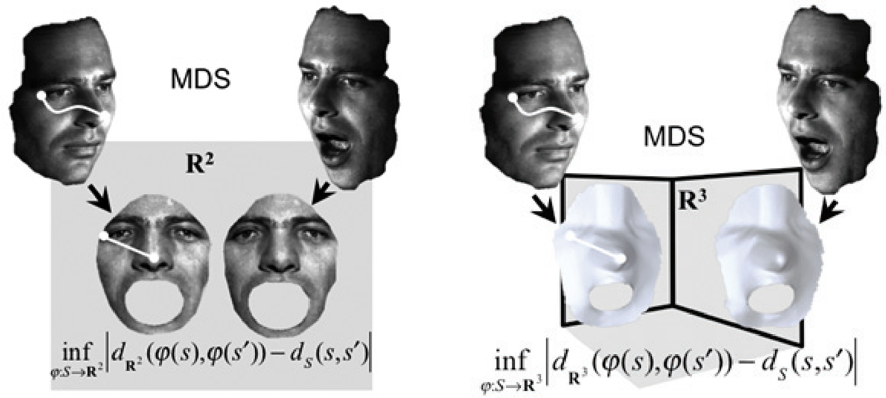

Figure 4. *Embedding to Euclidean space via multidimensional scaling.*

Here the distance can be interpreted as *isometrically* mapping S and Q into a space Z. The Hausdorff distance between the two mappings is then computed in Z. The optimization is over the mappings σ and ρ and the space Z. Again, scanning through all possible target spaces seems to be hard to compute.

In the early 2000s, we fixed the target space Z in an attempt to make some computational sense of related measures. When distances in Z can be computed analytically, as in $\mathbf{R}^n$, such a mapping belongs to the family of multi-dimensional scaling, or MDS, methods. MDS methods are intimately related to principle component analysis, which can be computed by singular value decomposition of the inter-geodesic distance matrix. Using such a method, we built a prototype face recognition system.

The next step involved changing the target space from $\mathbf{R}^n$ to Q, where distances are available numerically, and no longer analytically. This allowed for a consistent approximation of the Gromov–Hausdorff distance. The result is an inherently non-convex problem. Still, efficient initializations that exploit the smooth structure of low-dimensional geometric problems allowed us to transform the MDS approach into generalized MDS, or GMDS. With GMDS, we numerically approximate geodesic distances on Q, and apply convex optimization techniques to solve the mapping problem. Equipped with such a powerful tool, we have explored intrinsic symmetries, compared various surfaces, and even experimented with alternative distance measures, including an affine invariant metric defined on the surface and diffusion distances.

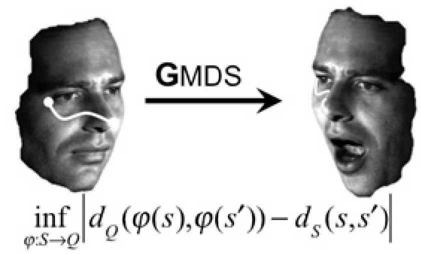

Figure 5. *Embedding into curved domains, where distances are evaluated numerically via generalized multi-dimensional scaling.*

With theoretical support from Facundo Memoli and Guillermo Sapiro that made it possible to treat sampled surfaces as metric spaces, we were able to show the Gromov–Hausdorff distance to be defined as three coupled GMDS problems. More importantly, we realized that d_{GH} can be viewed as the motivation for GMDS, which has become a powerful tool for the analysis of non-rigid shapes.


Figure 6. *Computing intrinsic symmetries.*

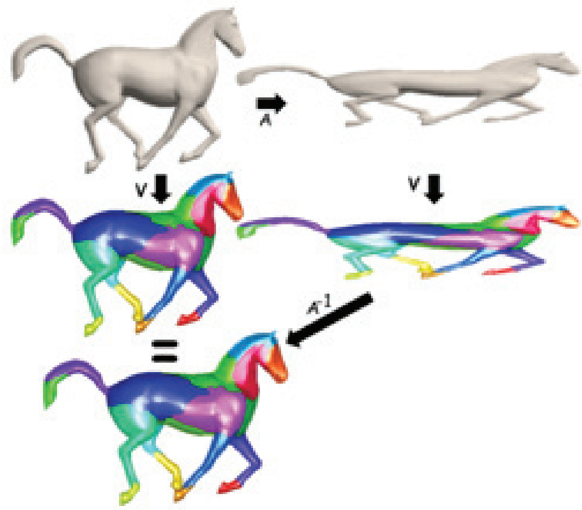
Figure 7. *Computing an affine invariant Voronoi diagram that could be used as an alternative distance within dGH.*

Co-authors of related papers published by the author include Alex Bronstein (TAU), Michael Bronstein (USI), Freddy Bruckstein (Technion), Yohai Devir and Anastasia Dubrovina (Technion), Asi Elad, Nahum Kiryati (TAU), Dan Raviv (Technion), Guy Rosman (Technion), Guillermo Sapiro (UMN), Nir Sochen (TAU), Irad Yavneh (Technion), and Gil Zigelman.

Readers can find a reference list in Numerical Geometry of Non-rigid Shapes *(Bronstein, Bronstein, and Kimmel, Springer, 2008) and on the author's website (www.cs.technion.ac.il/~ron).*

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