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Pattern Formation in Compressed Elastic Sheets

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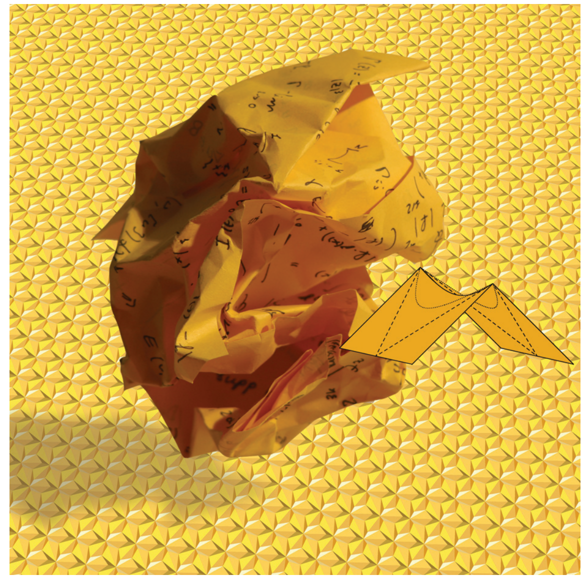

In origami a sheet of paper is folded along many segments, chosen carefully so that the paper approximates some given shape. Can any possible shape be approximated in that way? If the starting sheet of paper is large enough and a sufficiently high number of folds is admissible, the answer is affirmative. If the domain (the sheet of paper) is large enough, the aimed-for shape can be reached with a one-Lipschitz map. Hence, to answer the question, it suffices to show that "origami maps" are dense in the class of one-Lipschitz maps, with respect to uniform convergence.

The standard part of the strategy is to pass first to a C^1 isometry via the Nash–Kuiper theorem, then to a piecewise affine, one-Lipschitz map with small jumps in the gradient via linear interpolation. The final step consists of an explicit construction in which the grid is refined and each element "bulges up" to make the map isometric. For piecewise affine maps on a triangular grid, to guarantee that the map is isometric in each grid element it suffices to ensure that the lengths of the sides are conserved.

Although practical realizability of these maps is in most cases questionable, this result has important consequences for the study of the energetics of thin elastic sheets under compression, and in particular of the scaling of the needed energy in terms of the film thickness h, the natural small parameter in the problem. A specific construction makes it possible to approximate any sharp fold by a regularized fold with energy proportional to $h^{5/3}$. Replacing each fold of the origami map by this regularized fold, one can show that any possible shape can be reproduced with energy small with respect to h^{α}, for all $\alpha < 5/3$. This exponent matches experimental findings for crumpled sheets of paper.

Elastic sheets become substantially more rigid if Dirichlet boundary conditions are imposed, as practically relevant in, for example, the study of thin film blistering. In the last few decades, a number of different models have been proposed for studying this phenomenon; most are built on the Föppl–von Kármán plate theory. The discussion and comparison of several different approaches and their predictions in the talk at ICIAM show, in particular, that the optimal scaling of the energy is linear in the film thickness h.

For Further Reading

H. Ben Belgacem, S. Conti, A. DeSimone, and S. Müller, *Energy scaling of compressed elastic films*, Arch. Rat. Mech. Anal., 164 (2002), 1–37.

S. Conti, A. DeSimone, and S. Müller, *Self-similar folding patterns and energy scaling in compressed elastic sheets*, Comp. Meth. Appl. Mech. Eng., 194 (2005), 2534–2549.

S. Conti and F. Maggi, *Confining thin elastic sheets and folding paper*, Arch. Rat. Mech. Anal., 187 (2008), 1–48.

Sergio Conti is a professor of mathematics at the University of Bonn. This article is based on the invited talk he gave at ICIAM 2011.