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The Mathematics Behind *Sherlock Holmes: A Game of Shadows*

By Alain Goriely and Derek E. Moulton

During the summer of 2010, OCCAM (the Oxford Centre for Collaborative Applied Mathematics) received a call from Warner Bros. The mission, should we choose to accept it, was to help with the mathematical aspects of the studio's movie *Sherlock Holmes: A Game of Shadows*, in which Holmes's archenemy is the mathematician Professor James Moriarty. Initially, our task was to design the equations that would appear on a giant board in Moriarty's office. They were to be accurate for the time (around 1890) and potentially revealing of some of Moriarty's evil plans.

Holmes (Robert Downey Jr.) and Moriarty (Jared Harris) meet for the first time in Sherlock Holmes: A Game of Shadows.

Our task soon grew from simply designing equations to devising a secret code, creating the lecture Moriarty would give on a European tour, and providing suggestions on how Moriarty would use mathematics to carry out his plots, and how Holmes, in turn, would decipher them. Not surprisingly, very little mathematics actually made it to the big screen. Nevertheless, interesting snippets did make the final cut if you know where to look. In particular, a close examination of Moriarty's giant board reveals the whole mathematical story.

Arthur Conan Doyle actually gave precious little information on Moriarty, although Holmes does describe him as "a mathematical genius" and "the Napoleon of crime." On the academic side, we know that he wrote *A Treatise on the Binomial Theorem* and that "On the strength of it, he won the mathematical chair at one of our smaller universities" (Holmes, *The Final Problem*). (Although the name of the university is never revealed, eminent scholars at our institution have assured us that it must have been a university located somewhere on the river Cam.) In *The Valley of Fear*, Moriarty is said to have written another book, of which Holmes says, "Is he not the celebrated author of *The Dynamics of an Asteroid*, a book which ascends to such rarefied heights of pure mathematics that it is said that there was no man in the scientific press capable of criticising it?" Taking these cues, we delved into the mathematical mind of Moriarty and the fascinating mathematics of the turn of the 20th century.

The Code

One of the primary elements on the board, and a key aspect of Moriarty's evil plans, is the secret code that he uses to communicate with his henchmen as well as to encrypt his own information about his global empire. Given his obsession with the binomial theorem, we based the code we created for him on Pascal's triangle. The code has three elements: a public key, a coded formula, and a cipher. To code information, Moriarty first pieces together his message with letters taken from different locations in a horticultural book that he keeps in his office. Each letter of the message corresponds to three two-digit numbers: the page, line, and character number. By this process, the actual message is converted into a structured sequence of two-digit numbers: the *book sequence*. Moriarty further encodes the book sequence using Pascal's triangle.

The message requires a public key, an integer p. From each number p, one can build a sequence of numbers, the Fibonacci-p numbers, denoted F_p. The sequence is defined by $F_p(n) = F_p(n-1) + F_p(n-p-1)$, with $F_p(0) = 1$ and $F_p(n) = 0$, $n < 0$, and can be created by summing along the pth diagonal of Pascal's triangle. For $p = 0$, we recover the powers of 2 (horizontal lines on the Pascal triangle), whereas $p = 1$ corresponds to the classic Fibonacci sequence (first diagonal in the Pascal triangle).

Once p has been chosen, one can represent any two-digit number by giving the positions of the Fibonacci-p numbers that add to the given number in the minimal representation; that is, for any integer N, there is a unique representation $N = F_p(n) + \phi$, with $\phi < F_p(n-p)$. By giving the positions of the minimal representation, Moriarty converts the book sequence into a new, fully coded sequence of numbers.

As an example, suppose that the public key is $p = 3$. The Fibonacci-3 numbers are 1 2 3 4 5 7 10 14 19 26 36 50 . . . . Suppose that one line of the actual message is converted into the book sequence by taking characters from page 23 of the book: from line 10, characters 10, 5, 3, and 20; from line 17, characters 4, 18, 33, and 12. The "book sequence" would be:

23 10 10 05 03 20
23 17 04 18 33 12

To code, for instance, the number 23 with the Fibonacci-3 numbers, we note that the 4th and 9th digits of the Fibonacci-3 numbers are 4 and 19, which add to 23. Hence, 23 is coded as 0409. Following this rule, the fully coded sequence is

0409 07 07 05 03 0109
0409 0308 04 0408 0610 0207

Moriarty does two different types of encoding: He encodes messages that he passes to his associates during book signings, and he encodes information about his empire in his little red notebook. The code described above is used for both, with the difference occurring in the public key.

For a message to his associates, Moriarty must pass them the public key so that they know which Fibonacci number to use in decoding the message. This integer is included at a key moment in Moriarty's lecture, through a change in the value of a particular variable.

Cracking the code. Mathematics is key. Holmes first observes the work related to Pascal's triangle and the Fibonacci-*p* numbers written on Moriarty's board in his office. Later, Holmes notices a slight difference in Moriarty's lectures, guiding him to the idea that an integer key is being passed to an associate; Holmes eventually realizes that the key corresponds to a particular *p* in the Fibonacci-*p* numbers. He deduces that the horticultural book he saw in Moriarty's office serves as a cipher, based on the fact that the flower in Moriarty's office is dying.(We suggested, with no luck, that the dying flower be a sunflower head, which would have given a nice connection between Fibonacci numbers and phyllotaxis.) His powerful intellect does the rest.

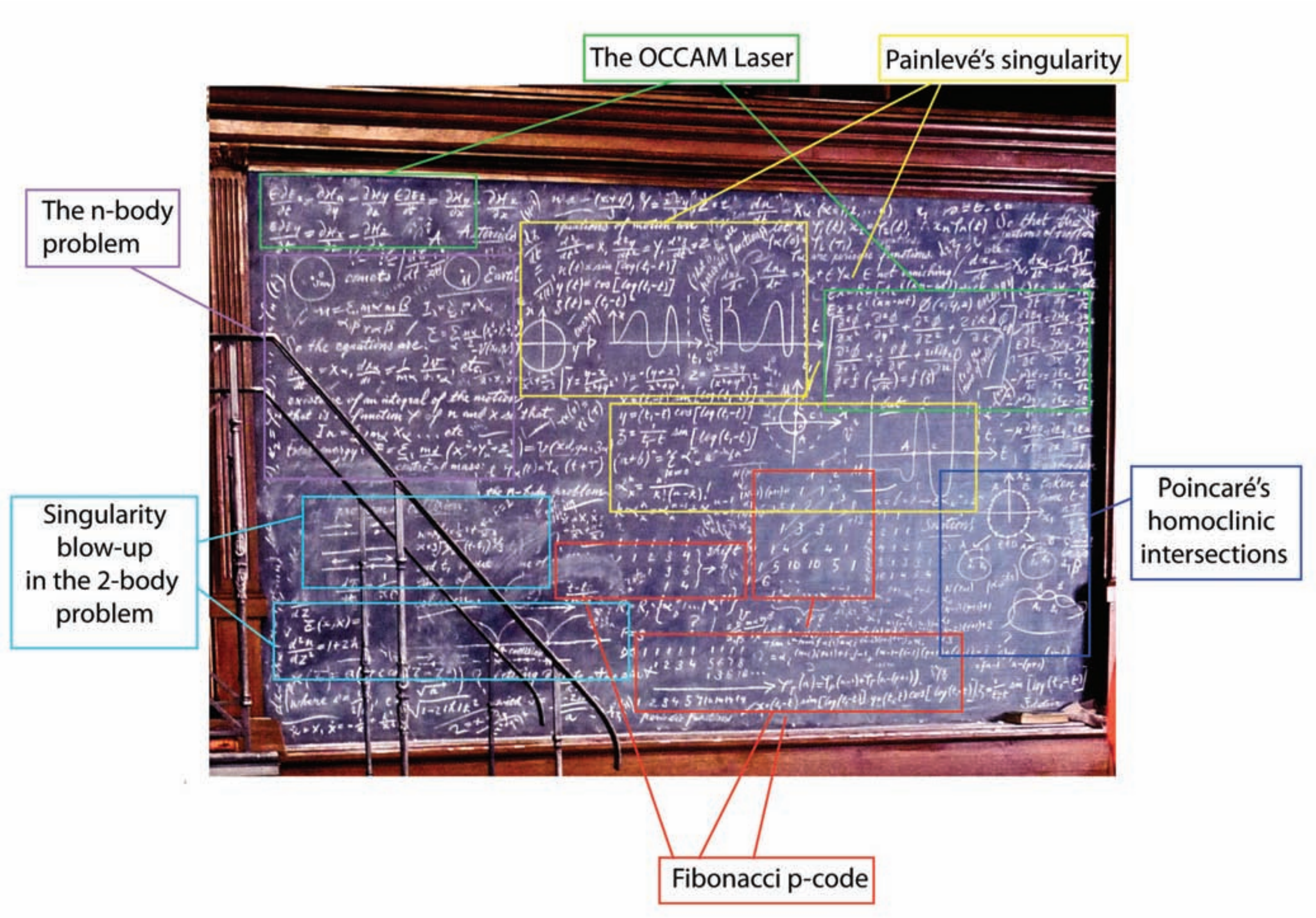
The Lecture

The other key mathematical element in the story is Moriarty's lecture tour. Our goal was to design a lecture that he could have plausibly given around 1895; the topic needed to fit with his work on the dynamics of asteroids, and to be important enough to warrant a European tour. While on the surface the lecture tour is a vehicle for Moriarty to oversee and orchestrate his evil empire, we decided that the lecture itself should be explosive and characterize the villain and his motivation for crime.

For inspiration, we turned to three major late-19th-century works on celestial mechanics. The first was George Hill's solution to the so-called restricted three-body problem (1878), which gives the motion of the moon (or, say, an asteroid) moving around the earth–sun.

Our second stepping stone was the work of Henri Poincaré on the *n*-body problem. The *n*-body problem consists of finding solutions of the Newton equation for the gravitational interaction of *n* masses. The case $n = 2$ is the classic Kepler problem that leads to elliptical orbits. The problem was considered so important that Oscar II, King of Sweden and Norway, promised a special prize for the mathematician who could solve it. The winner of the prize was Poincaré, who expanded his work on the subject into a series of three books (*New Methods of Celestial Mechanics*) published in 1892. These books are regarded as some of the most influential works of the early 20th century and are usually credited as the origin of such important mathematical concepts as geometric analysis for dynamical systems, chaos, and asymptotic expansions. In the initial edition, Poincaré made a substantial mistake (which ultimately led to the discovery of sensitivity to initial conditions and chaos in mathematics).

We conjecture that the awarding of the prize to Poincaré is very upsetting to Moriarty, as the version published in his own (unintelligible) book is in his view the correct solution to the problem. We were interested to note that at the time, the leading scientific figures did not hesitate to criticise



Deconstructing Moriarty's blackboard. All the mathematics behind the plot is present on the board, but a true evil genius must have a few ideas on the back burner as well. Aside from the code and the lecture, both of primary importance to the story, Moriarty was developing equations for a laser. These equations, based on Maxwell's equations and contributed by Oxford's John Ockendon and the OCCAM team, can also be seen on the board.

each other publicly in vitriolic terms that would certainly be deemed inappropriate in our modern society. (For instance, Poincaré writes, "A great number [of Gyldén's results] are clearly false; most of them are given in a way which is too obscure to decide whether they are true or false."*) Moriarty, who is obviously lacking in honesty, would have no qualms about ranting against French and American mathematicians.

Our third important source for Moriarty's lecture was the work of Paul Painlevé. In 1895 Painlevé was invited to Stockholm (again by King Oscar II) to deliver a series of lectures on his work. (Poincaré was originally invited as well but could not attend.) The event was considered so important that King Oscar himself attended the opening lecture. Of relevance for Moriarty's lecture is Painlevé's work on collisions. Painlevé looked at the possibility of collisions between masses in gravitational interactions and proved some fundamental results complementing the work of Poincaré (in particular, he proved the non-existence of non-collisional singularities for the three-body problem). He also showed that both Weierstrass and Poincaré had missed an important class of solutions for colliding planets.

Also of historical importance is that Painlevé would become the minister of war (and later of aviation) during the Great War, an interesting connection with Moriarty's obsession with weaponry. In many ways, Painlevé is the man that Moriarty could have become if he had been properly recognised and had not developed his "hereditary tendencies of the most diabolical kind." The lectures Painlevé gave in 1895 were typed and published. However, the very last chapter, on celestial mechanics, was never typeset, and the original handwritten manuscript provided us with useful information on the calligraphy of the time.

Unraveling it all. With the villain and his mathematics firmly established in our minds, we produced pages of possible lecture material, secret codes, and evil plots, complete with logical deductions used by Holmes to unravel it all. (We even tried our hand at a bit of scriptwriting, which almost surely never made it off our contact's desk.) We passed everything on to Warner Bros., who were kind enough to make a donation to OCCAM. We were never in it for money, though—the chance to communicate some real mathematics on the big screen was all the motivation we needed.

Epilogue

In December 2010, we were invited to be on set at Hatfield House, near London, for the filming of the office scene, where, as far as we were concerned, the central character was to be the blackboard. The day was cold and miserable as only English winter days can be. After waiting patiently for many hours, we finally got to see the board. It was beautiful, imposing, and full of typos.† We spent countless hours helping the professional calligrapher rectify the mistakes on the board and teaching the crew about the subtleties of subscripts and curly derivatives, our own contribution to outreach. The art department of Warner Bros. has an amazing team dedicated to making sure that the finest details are accurate.

But by the end of the day, two simple facts of life became obvious to us: First, on the totem pole of Hollywood, mathematicians sit comfortably at the very bottom, and second, the long and uneventful hours of waiting and preparation made us realize how charming and pleasant our office life truly is: There is really no business like academic business.

A year later, the movie came out and we were invited to the screening in London. The movie is rather long and violent (a fast-paced action movie, in Hollywood terms). Clearly, this new Sherlock Holmes is more muscular and physical than any of his predecessors, something like a Victorian John McClane capable of sustaining unlimited beating, as *Die Hard* fans will know. His eventual victory is based as much on his usual deductive powers as on an uncanny ability to predict unreasonably well the combined outcome of many random events, all in slow motion. This is rather unfortunate: Holmes's timeless enduring quality is not an unphysical skill of predicting where his opponent will punch twenty moves ahead. His real skills are the ability to use his broad scientific knowledge, his logical mind, and his perseverance to piece unrelated information into a single unified picture in order to crack intellectual puzzles. Indeed, Sherlock Holmes would have made a fine applied mathematician.

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*A delightful account of these exchanges can be found in June Barrow-Green's book *Poincaré and the Three-body Problem* (AMS/LMS, 1997).

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†In case you have any doubt, ask somebody with no mathematical training to copy formulas for you.

Alain Goriely, professor of mathematical modelling at Oxford University, is director of the Oxford Centre for Collaborative Applied Mathematics. Derek Moulton is a postdoctoral research assistant at OCCAM.