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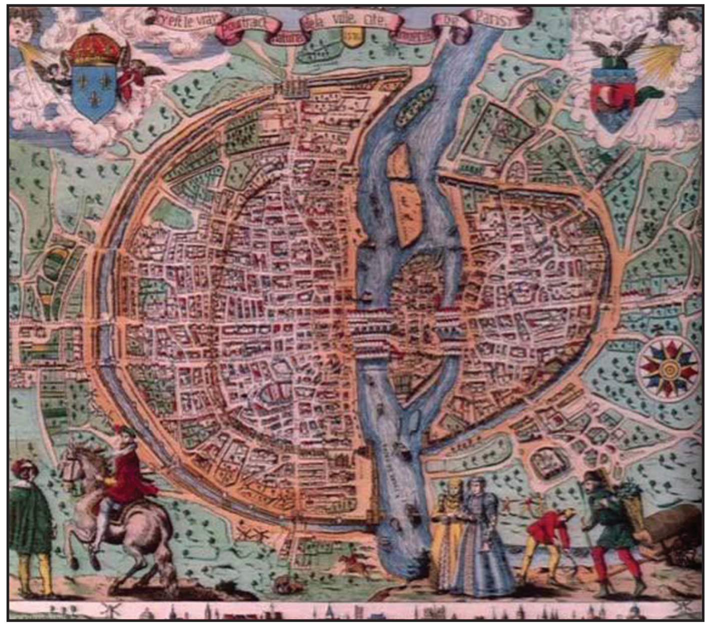
Unprecedented Simplicity

An elegant proof of the optimality of the semicircle capped Richard Tapia's talk on the isoperimetric problem in Minneapolis.

By James Case

Revisiting the isoperimetric problem in an invited address at the 2012 SIAM Annual Meeting, Richard Tapia of Rice University observed that the earliest known proofs of the "isoperimetric theorem," including the analytic one given by Euler (in 1744) and the geometric one offered by Jakob Steiner (in 1838), were incomplete for having assumed without proof that a solution exists.

The roots of the problem, Tapia said, are lost in antiquity. Pythagoras seems to have been aware—before 500 BCE—that of all simple closed planar curves of a given length, the circle encloses the greatest area, although the remarks attributed to him on the subject are somewhat ambiguous. Euclid (330–260 BCE) proved that the equilateral triangle maximizes the enclosed area among all triangles of given perimeter, and that the square does likewise among all such rectangles. In a book titled *On Isoperimetric Figures*, Zenodorus (200–140 BCE) reportedly gave proofs of the following two theorems:



Emulating Queen Dido's application: Medieval Paris, like many other cities of the age, was enclosed in semicircular walls, with the openings closed off by a river.

Among all polygons of equal perimeters and equally many sides, the regular polygon contains the greatest area.

The circle encloses a greater area than any regular polygon of equal perimeter.

The book itself has been lost. We know of it only from partially preserved contents in the works of Pappus (290–350 CE) and Theon (335–405 CE) of Alexandria.

Despite the lack of a formal proof, the validity of the isoperimetric theorem was widely accepted in the ancient world. Virgil's *Aeneid* recounts the story of Queen Dido's clever application of the theorem. When the ship in which she was travelling washed up in a storm on a foreign shore, she persuaded the local monarch to grant her as much land as she could cover with the hide of a bull. Dido instructed her servants to cut the hide into a single long thong (best accomplished by starting on the perimeter and spiraling inward), and then to string it out in the shape of a semicircle stretching inland from the seashore. Though surprised by the result, the monarch stood by his bargain, and Dido founded within the ensuing land grant what eventually became the thriving city of Carthage. Tapia exhibited pictures of several medieval cities, including Paris, whose protective walls formed semicircles bounded on the open side by unfordable rivers.

Nothing more of mathematical consequence was said on the isoperimetric problem until 1744, when Euler cast it in a form essentially equivalent to the following "Queen Dido" problem:

$$\begin{aligned}&\text{maximize} \int_{-a}^{a} y(x)\,dx\\&\text{subject to} \int_{-a}^{a} \sqrt{1+\left[y'(x)\right]^2}\,dx = a\pi;\\&y(-a) = y(a) = 0,\end{aligned}$$

and concluded that the solution—which he tacitly assumed to exist—can only be the semicircle $y_o(x) = \sqrt{a^2 - x^2}$, $-a \le x \le a$. In 1755, at the end of a brief letter to Euler, the then 19-year-old Lagrange attached an appendix describing his revolutionary method of "variations," whereby he could readily deduce "Euler's equation" along with the rudimentary "multiplier rule" used by the master to obtain his solution. So taken was Euler with this new method that he abandoned his own geometric techniques, adopted those of Lagrange, coined the term "calculus of variations," and referred thereafter to multiplier theory as "Lagrange multiplier theory." Tapia noted, in a technical aside, that the arc length integral for the semicircle does in fact exist as an improper integral.

The next significant development came from Steiner (1796–1867), a Swiss geometer who disliked analysis and doubted that anything proved with it couldn't be proved with traditional geometry. To strengthen his case, he offered (in 1838) a novel geometric proof of the isoperimetric theorem. Analysts of the time, however, led by Dirichlet, were quick to point out that the proof was incomplete, as it rested on the assumption that a solution exists. What would be required, they wondered, to "complete Steiner's proof"?

The question remained open until 1879, when Weierstrass developed his famous sufficiency theory and applied it to the isoperimetric problem. Over the next forty odd years, a substantial series of papers by a who's who of prominent analysts offered alternative proofs of the optimality of the semicircle $y_o(x)$.

Tapia argues, however, that all the early sufficiency proofs were unnecessarily complicated: The authors had failed to recognize and exploit the underlying concavity of the problem. Indeed, he noted, it was not until Johan Jensen (in 1905) and Herman Minkowski (circa 1907) that modern mathematics began to recognize the importance of functional convexity, despite the fact that both Archimedes and Fermat had defined and exploited the property of convexity for geometric curves.

Tapia's first sufficiency proof involves the functional

$$J(y) = \int_{-a}^{a} \left\{ y(x) - a\sqrt{1 + \left[y'(x)\right]^2} \right\} dx;$$

the semicircle $y_o(x) = \sqrt{a^2 - x^2}$, $-a \le x \le a$; and a concave alternative $y(x) \ne y_o(x)$ satisfying the boundary condition $y(-a) = y(a) = 0$. Defining $\eta = \eta(x) = y(x) - y_o(x)$ and $\varphi(t) = J(y_o + t\eta)$, he combines the facts that $\varphi'(0) = 0$ and $\varphi''(0) < 0$ with Taylor's theorem in the form $\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(\theta)$ for some $\theta \in (0,1)$ to conclude that $J(y) < J(y_o)$ and, hence, that among functions of the same arc length, y_o is indeed the desired solution of the isoperimetric problem.

Hard though it is to imagine a simpler proof of this or any other significant result, Tapia offered a variety of comparably elementary variations, all of which are independent of the means by which one arrives—as did the ancient Greeks—at the hypothesis that the semicircle solves Dido's problem. Even Peter Lax's clever proof [1] of the isoperimetric theorem seems—as Tapia emphasized with a set of patently photoshopped slides depicting Lax and himself as matadors competing mano à mano for the title of author of the simplest proof—unduly complicated in comparison.

Why, Tapia wondered aloud, did neither Euler nor Lagrange, nor any other 19th-century analyst, think to use Taylor's theorem as a means of proving the isoperimetric theorem? Euler was certainly familiar with Taylor's theorem, if not with the particular form of the remainder used by Tapia in the proof presented here. On the other hand, the form in question is the work of Lagrange, and virtually every subsequent analyst has employed it on occasion.

In conclusion, Tapia voiced his belief that the isoperimetric problem has been the most impactful of all the problems of antiquity on the development of modern mathematics. Few if any questions have occupied the attention of so many giants in the field, or inspired them to greater heights of creativity, he said. Though cases could be made for any of the ancient construction problems (doubling the cube, trisecting the angle, squaring the circle, and perhaps a few others), Tapia's candidate seems as worthy as any.

References

[1] P.D. Lax, *A short path to the shortest path*, Amer. Math. Monthly, 102:2 (1995), 158–159.

James Case writes from Baltimore, Maryland.