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Celestial Mechanics Theory Meets the Nitty-Gritty of Trajectory Design

Capture Dynamics and Chaotic Motions in Celestial Mechanics: With Applications to the Construction of Low Energy Transfers. *By Edward Belbruno, Princeton University Press, Princeton, New Jersey, 2004, xvii + 211 pages, $49.95.*

BOOK REVIEW

By James Case

In January 1990, ISAS, Japan's space research institute, launched a small spacecraft into low-earth orbit. There it separated into two even smaller craft, known as MUSES-A and MUSES-B. The plan was to send B into orbit around the moon, leaving its slightly larger twin A behind in earth orbit as a communications relay. When B malfunctioned, however, mission control wondered whether A might not be sent in its stead. It was less than obvious that this could be done, since A was neither designed nor equipped for such a trip, and appeared to carry insufficient fuel. Yet the hope was that, by utilizing a trajectory more energy-efficient than the one planned for B, A might still reach the moon.

Such a mission would have seemed impossible before 1986, the year Edward Belbruno discovered a class of surprisingly fuel-efficient earth–moon trajectories. When MUSES-B malfunctioned, Belbruno was ready and able to tailor such a trajectory to the needs of MUSES-A. This he did, in collaboration with J. Miller of the Jet Propulsion Laboratory, in June 1990. As a result, MUSES-A (renamed Hiten) left earth orbit on April 24, 1991, and settled into moon orbit on October 2 of that year. Without the assistance of Belbruno and Miller, Japan might not have become the third nation in history to send a spacecraft to the moon. After performing a series of maneuvers and scientific experiments, Hiten was purposely crashed onto the lunar surface on April 10, 1993.

N-Body Problems Revisited

Belbruno and Miller modeled Hiten as a particle P of negligible mass moving under the influence of gravitational forces from the earth (E), the sun (S), and the moon (M). Before getting close to M, P is only slightly affected by it, suggesting comparison with the three-body problem involving only E, S, and P. In the immediate neighborhood of M, the forces exerted on P by E and M far exceed that exerted by S, inviting consideration of a second three-body problem involving only P, E, and M. Accordingly, Belbruno devotes an entire chapter of his three-chapter book to a review of N-body problems, with special attention to small values of N.

The classic N-body problem concerns the motion of N mutually gravitating point masses $P_1, P_2, \ldots, P_N$ of magnitude $m_1 \geq m_2 \geq \ldots \geq m_N$ in 3-space. Although the case $N = 2$ (aka the Kepler problem) is completely soluble in closed form, it remains useful to distinguish certain special cases. Among the most frequently encountered are circular Kepler motion, in which both bodies move in concentric circles about a common center of mass, and elliptic Kepler motion, in which each body traces out an ellipse focused at the common center of mass. If one of the bodies far outweighs the other, as the sun does the earth, the more massive will appear fixed and the other will seem to gyrate around it.

Belbruno describes the various coordinate systems appropriate to the planar three-body problem, and points out that the so-called Jacobi coordinates are particularly well suited to the study of three-body problems in which the least massive body is "restricted" to be of negligible mass. In that case it is often convenient to choose mass units in which $m_1 + m_2 = 1$, and to write $m_2 = \mu \leq 1 - \mu = m_1$, which implies $0 \leq \mu \leq ½$. When $m_3 = 0$, the equations of motion separate into an autonomous pair of second-order ordinary differential equations, which can be solved explicitly for the motion of the more massive bodies about their common center of mass, and a non-autonomous pair involving the solutions of the first two and describing the motion of the massless third body relative to that same center of mass.

In the special case in which the two massive bodies move in concentric circles about their common center of mass, it is possible to introduce a system of rotating coordinates in which one of the axes coincides with the line joining the two massive bodies. In such coordinates, the second set of equations assumes the autonomous form

$$\ddot{x} + 2A\dot{x} = f(x) + g(x), \tag{1}$$

in which

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and in which $f(x) = x = (x_1, x_2)^t$ is the familiar centrifugal force vector and $g(x)$ represents the gradient of an appropriately defined potential function. The points x at which f and g both vanish are known as the Lagrange points $L_1, \ldots, L_5$ because Euler—who found them before Lagrange was born—already has too many things named after him. L_1–L_3 lie on the line joining the two massive bodies; L_4 and L_5 are symmetrically placed on opposite sides of that line. During the 1960s, L_1–L_3 were shown to be unstable, and L_4 and L_5 to be stable.

Belbruno then demonstrates that a certain function $J(x,\dot{x})$—commonly known as the Jacobi energy function—is a constant of the motions (1), making the sets $J^{-1}(C) = \{(x,\dot{x}) \in R^4 \mid J = C\}$ invariant under those motions. By evaluating the quantities $C_k = J(L_k,0)$ at each of the Lagrange points L_k, $k = 1,2,3,4,5$, one obtains an energy scale $C_4 = C_5 < C_3 < C_1 < C_2$ of particular significance for the planar restricted three-body problem.

The projections of the manifolds $J^{-1}(C)$ onto the x_1,x_2-plane are known as Hill's regions. Their shapes vary with C, more or less as indicated in parts (a)–(e) of Figure 1. They are significant because P_3 is necessarily contained in one of them—it cannot enter the shaded regions.

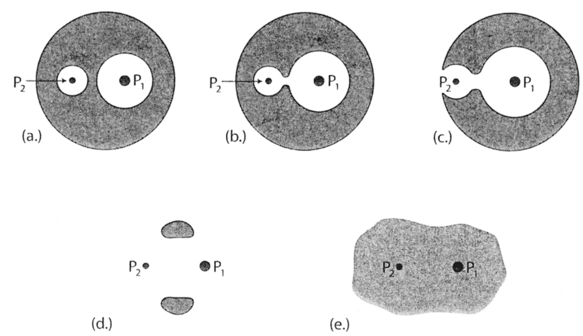

Figure 1. *Hill's regions for various values of the parameter C. The massless particle P_3 can lie anywhere in the unshaded portions of the x_1x_2-plane but cannot enter the shaded parts. Figures from* Capture Dynamics and Chaotic Motions in Celestial Mechanics.

When $C > C_2$, for instance, P_3 cannot pass from the unshaded region containing P_1 to that containing P_2, or vice versa. Nor can it pass from either of those regions to the unshaded exterior. Until $C < C_2$, P_3 cannot hope to escape from whichever unshaded region originally contained it. When $C_1 < C < C_2$, however, a passage opens between the unshaded regions surrounding P_1 and P_2, permitting P_3 to pass from one to the other. Yet it still cannot pass between the interior and the exterior unshaded regions. Only when $C < C_1$ does P_3 become free to roam between those two regions as well. The unshaded region remains connected until $C < C_3$, then breaks apart into an upper and a lower component when $C_4 < C < C_3$, before vanishing entirely when $C < C_4$. Without the ability to move between $P_1 = E$ and $P_2 = M$, there can be no earth–moon trajectories for $P_3 = P$.

"Ballistic" Capture

To maneuver a spacecraft into orbit around the moon, one must arrange for M's gravitational field to "capture" P. It is far from clear, however, what "capture" means in this context. Permanent capture in the three-dimensional N-body problem has traditionally been taken to mean that all the distances r_{ij} between point masses P_i and P_j remain bounded as $t \to \infty$, while at least one such distance grows without bound as $t \to -\infty$.

Permanent capture is difficult to achieve, even in the case $N = 3$. Indeed, it was proved in 1918 that the set of initial positions and momenta leading to it constitute a set of measure zero in phase space. Proof that the set is non-empty came only in 1960, when K.A. Sitnikov considered a special case of the elliptic restricted three-body problem depicted in Figure 2. What is now known as the Sitnikov problem requires that $\mu = ½$ and that P_3 be constrained to lie on the vertical (Q_3) axis. Such constraint is possible because the vertical axis is invariant when $\mu = ½$. Sitnikov then proved that $Q_3(t)$ remains bounded for $t > 0$ if and only if $|Q_3(0)| + |\dot{Q}_3(0)|$ is sufficiently small.

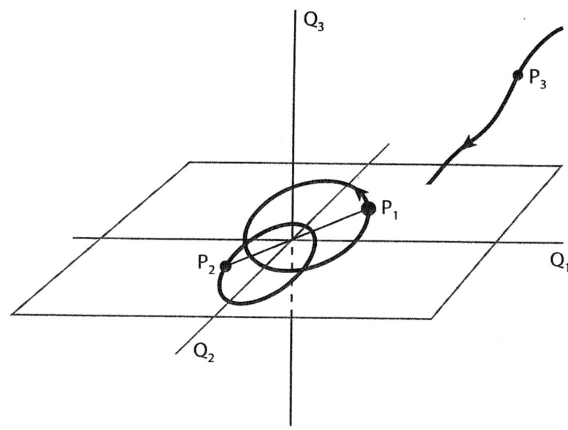

Figure 2. *In the three-dimensional elliptic restricted three-body problem, the massive particles P_1 and P_2 des-cribe elliptic orbits with a common focus at the origin of coordinates, undisturbed by the approach of the massless P_3.*

Motions for which $|Q_3(t)| \to \infty$ and $|\dot{Q}_3(t)| \to 0$ as $t \to \infty$ are known as parabolic orbits, and can be regarded as "critical" escape orbits. Sitnikov also demonstrated the existence of trajectories for which $\limsup_{t=0} |Q_3(t)| = \infty$ and $\liminf_{t=0} |Q_3(t)| = 0$. V.M. Alekseev and others subsequently demonstrated the existence of a neighborhood of the parabolic orbits in trajectory space in which the motion is chaotic.

Recognizing that permanent capture is too much to hope for in lunar trajectory design, Belbruno introduces a weaker concept called "ballistic capture." He declares P to be ballistically captured by M's gravitation if P approaches M in such a way that their combined total energy—kinetic plus gravitational potential—is negative. For then, were they alone together in space, P would enter a (bounded) elliptic orbit, rather than a (boundless) hyperbolic one, about M. Practically speaking, this means that P must carry enough fuel to reduce its speed relative to M—presumably by firing its retro rockets on arrival at the desired distance from M—until its kinetic energy is exceeded by its gravitational potential at that distance. Although guidance rockets typically fire for seconds or even minutes, their effects are treated as instantaneous impulses for the purposes of trajectory design.

Impulsive Control in Lunar Mission Design

Lunar missions typically involve three impulses. The first propels P out of its initial (ordinarily circular) earth orbit into a more eccentric orbit, headed in the general direction of M; the second effects a mid-course correction, and the last slows P enough to permit capture. Unlike the simpler "Hohmann transfers" employed by previous lunar missions, the trajectory transfers discovered by Belbruno in 1986 require no third impulse.

The older class of transfers, discovered by W. Hohmann during the first quarter of the 20th century, involves (ideally) no mid-course correction. Whereas the initial impulse results in an almost parabolic (eccentricity ≈ 0.97) orbit around E, calculated by ignoring the moon's gravity, the final impulse, obtained by ignoring the earth's gravity, produces a far less eccentric elliptic orbit around M.

For $C = C_2$, Hill's regions about E and M osculate at L_2; for slightly smaller values of C, a narrow "neck" opens up between them, through which P can pass. For $C < C_2$, L_2 vanishes, bifurcating into a family of unstable periodic orbits known as Lyapunov orbits. Inside the neck, the projections of solutions of (1) onto x_1x_2-space become chaotic, as shown in Figure 3.

The intersection of two tubular invariant manifolds in $J^{-1}(C)$ includes a single Lyapunov orbit Λ, which forms a simple closed loop in 4-space. All the solutions on one of the tubes spiral in toward Λ, and all those in the other tube spiral away from it. Hence, in the immediate vicinity of Λ, a small but timely impulse imparted to P as it approaches Λ along an inwardly spiraling solution of (1) will transfer it onto an outwardly spiraling solution. If the inwardly spiraling solution originates near E, while the outwardly spiraling one is bound for M, the resulting compound trajectory connects E to M.

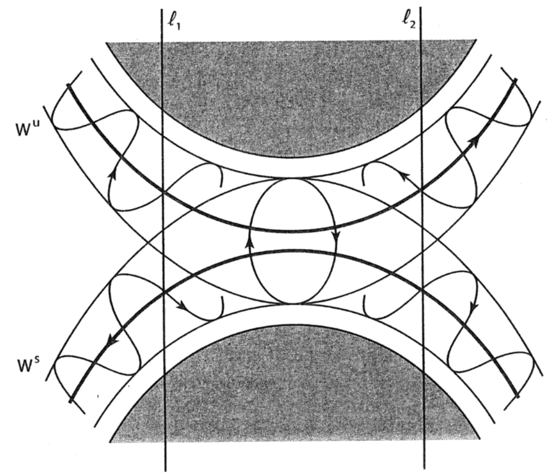

Figure 3. *The closed orbit in the "neck region" of the unshaded portion of the x_1x_2-plane is the limit cycle of the inwardly spiraling pair of solutions and, with time reversed, of the outwardly spiraling pair as well. A small impulse can thus divert P_3 from an inwardly spiraling course to an outwardly spiraling one, causing it to enter one side of the neck and exit the other.*

Belbruno calculated just such an earth–moon trajectory in 1986 (Figure 4). That trajectory would not have been of much use to the Japanese, since its initial distance from E far exceeds the requisite $r_E + 200$ km and the final distance from M far exceeds the desired $r_M + 100$ km. Here, r_E and r_M denote the mean radii of the earth and the moon. Belbruno's trajectory does, however, demonstrate the feasibility of navigating through the neck of Hill's region when $C < C_2$. Belbruno devotes a significant portion of his third chapter to detailed descriptions of this trajectory and of the more complicated one he and Miller designed for Hiten. Because the latter lies in $J^{-1}(C)$ for $C < C_1$, the shaded portion of Hill's region does not separate the plane into an interior and an exterior component. This made it possible for the spacecraft to venture almost four times as far from the earth as the moon ever does.

Belbruno's accomplishments are well known in the astrodynamical community, and many recent mission designs have been influenced by his work. He himself mentions the SMART1 lunar mission of the European Space Agency, a NASA mission to Jupiter's moon Europa, Japan's Lunar A mission, and a Japanese mission to Mars called Planet B. Jerrold Marsden was quick to acknowledge via e-mail that Belbruno's work on Hiten, along with Belbruno's joint work with Brian (no relation) Marsden on "resonance hopping," has influenced a lot of the work his own group has been doing lately, and suggests that more than a few others in the field are following his lead.

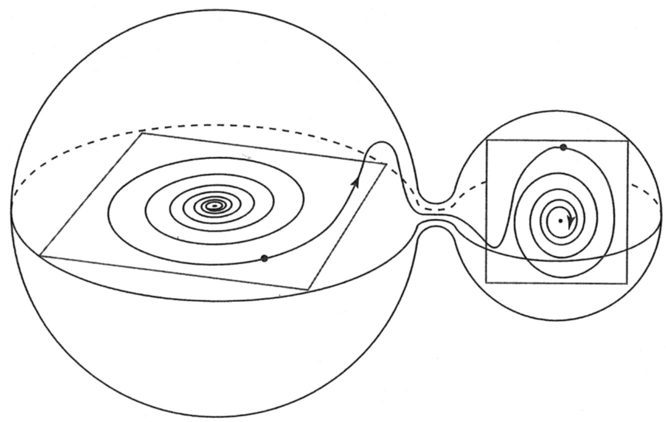

Figure 4. *Representation of a complete transition from an outwardly spiraling earth orbit, through the neck of a three-dimensional Hill's region, to an inwardly spiraling lunar orbit.*

Belbruno's book combines the latest theoretical results in celestial mechanics with the nitty-gritty of successful trajectory design. It is possible to gauge the breadth of its coverage by the extent of its bibliography, which contains 231 references to authors as diverse as Kepler, Newton, Euler, Lagrange, Hamilton, Poincaré, Sundman, Moulton, Morse, Moser, Milnor, J. Marsden, B. Marsden, Smale, Saari, and Xia. A truly startling array of mathematicians has contributed to our current (practical and theoretical) understanding of the N-body problem.

James Case writes from Baltimore, Maryland.