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Accurate Eigenvalues for Fast Trains

By Ilse C.F. Ipsen

Participants from Berlin, Bologna, and Basel arrived by train, as did those who had flown into Frankfurt or Düsseldorf from Madrid, Manchester, and Raleigh/Durham. The occasion: IWASEP5 and a workshop of the GAMM activity group in linear algebra†, both held in Hagen, Germany, during the week of June 28, 2004. The subjects: accurate solution of eigenvalue problems, and linear algebra in systems and control theory. One application: trains.*

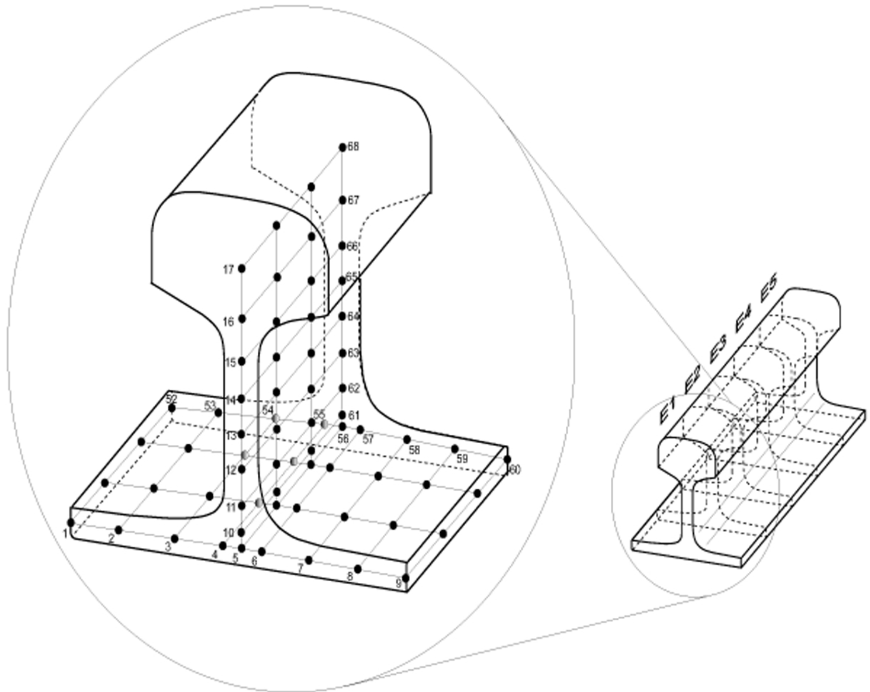

Figure 1. *Discretization of the rail.*

Vibration Analysis of Rails Excited by High-Speed Trains

With new ICE (Inter-City Express) trains crossing Europe at speeds of up to 300 kilometers/hour, sound and vibration levels in the trains are an important concern. To address the problem, Volker Mehrmann and Christian Mehl of the Technical University of Berlin, working with SFE, a structural engineering firm in Berlin, have studied the resonances of railroad tracks excited by high-speed trains.

The trains are modern, Mehrmann and Mehl explained in talks at the Hagen workshops, but the numerical methods used to design them are at least 30 years old. More often than not, classic finite element packages produce answers that fail to deliver even a single correct digit. Mehrmann and Mehl showed how modern methods from linear algebra can provide answers accurate to three digits in single(!) precision, with no change in the finite element model. Central to their idea is a careful exploitation of the structure of the eigenvalue problem.

An understanding of what Mehrmann, Mehl, and their collaborators have done starts with a model of the railroad track. If the rails are straight and infinitely long, a simple finite element discretization, as shown in Figure 1, produces an infinite-dimensional system of ordinary differential equations:

$$M\ddot{x} + D\ddot{x} + Kx = F,$$

where the matrices M, D, and K are block-tridiagonal. If the sections of track between crossties are identical, the system is also periodic. Application of a Fourier transform and combination of all unknowns located between crossties j and $j + 1$ into one vector y_j produce a three-term difference equation with constant coefficients:

$$A_1^T y_{j-1} + A_0 y_j + A_1 y_{j+1} = F_j,$$

where the complex coefficient matrices depend on the excitation frequency (and with the superscript T denoting the transpose). The matrix A_0 is symmetric ($A_0^T = A_0$); A_1 is singular. Setting $y_{j+1} = \kappa y_j$ leads to a rational eigenvalue problem:

* The fifth International Workshop on Accurate Solution of Eigenvalue Problems (http://www.fernuni-hagen.de/mathphys/iwasep5/); see Beresford Parlett's article on page 3.

†For information about GAMM (Gesellschaft für Angewandte Mathematik und Mechanik), the activity group in applied and numerical linear algebra, and the workshop, see http://www-public.tu-bs.de/~hfassben/gamm/gamm_fa_anla.html.

$$R(\kappa) = y = 0,$$

where

$$R(\kappa) \equiv \frac{1}{\kappa} A_1^T + A_0 + \kappa A_1.$$

Because R (κ) = $(1/\kappa)^T$, this eigenvalue problem is palindromic (named for verbal palindromes, such as: Was it a car or a cat I saw?). As a result, the eigenvalues occur in pairs (κ, $1/\kappa$). Such a spectrum is called "symplectic"; an example is shown in Figure 2. For an analysis of the track vibrations, all finite, non-zero eigenvalues and eigenvectors have to be computed for many frequencies in the 0–5000 Hz range.

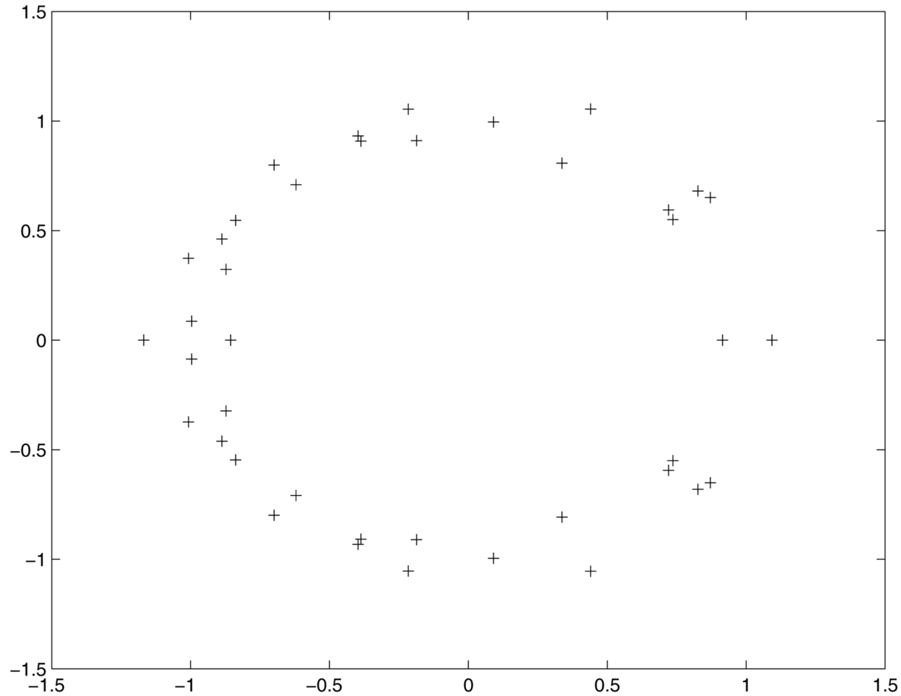

Figure 2. *Example of a symplectic spectrum.*

Eigenvalues

A popular approach to the rational eigenvalue problem $R(\kappa)y = 0$ is to convert it to a polynomial eigenvalue problem, which in turn is linearized. In this approach, $R(\kappa)y = 0$ is written as the polynomial eigenvalue problem

$$P(\lambda)y = 0,$$

where

$$P(\lambda) = \lambda^2 A_1^T + \lambda A_0 + A_1.$$

Application of a classic linearization $z = \lambda y$ leads to something like

$$\begin{pmatrix} 0 & I \\ -A_1 & -A_0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} I & 0 \\ 0 & A_1^T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

This is a generalized eigenvalue problem, which can be solved by public-domain software, such as the MATLAB function *eig*. That's it, problem solved! Right?

Not quite—in fact, nothing we have produced is useful. The two big matrices in the linearized problem are not symmetric; the structure of the original problem has been destroyed. This means that numerical methods are not going to deliver a symplectic spectrum—that is, the computed eigenvalues are not going to exhibit mirror symmetry, and computed eigenvalues at 0 and ∞ are not going to occur in pairs. We cannot trust any of the computed eigenvalues. The conventional methods, unable to recognize the structure in the original problem, have produced inaccurate, useless eigenvalues.

What can be done? Accurate computed eigenvalues should, at the very least, retain the mirror symmetry of a symplectic spectrum. If it is to have a chance at delivering a symplectic spectrum, a numerical method is much better off if it can work on a palindromic polynomial. What we need, in other words, is a way to linearize without losing the palindromic structure.

To this end, Mehrmann and Mehl, working with Niloufer and Steve Mackey, have developed a theory of structure-preserving linearizations that yields a whole vector space of linearizations. Not all the linearizations are useful, and choosing one that is "best" (whatever that means) is an open problem. For the application at hand, one that works is:

$$\left(\lambda Z + Z^T\right) \begin{pmatrix} z \\ y \end{pmatrix} = 0,$$

where

$$Z \equiv \begin{pmatrix} A_1^T & A_0 - A_1 \\ A_1^T & A_1^T \end{pmatrix}$$

which is a palindromic linearization‡ of $P(\lambda)$.

We are still not home free. As it turns out, the computed eigenvalues in this problem are badly scaled: Their magnitudes range from 10^{-15} to 10^{15}. This is so because many of the exact eigenvalues of $P(\lambda)$ are located at 0 and ∞. An essential step is thus a similarity transformation that removes (deflates) the eigenvalues at 0 and ∞, while managing to preserve the palindromic structure. Only then can the remaining eigenvalues of $P(\lambda)$ be computed, with a carefully customized Jacobi method.

The Upshot

It pays to preserve structure, if this can be done in a numerically viable fashion. In the train problem described here, structure-preserving linear algebra methods rescued an otherwise moribund computation. They made it possible to compute accurate answers—even in the face of a simplistic computational model and a coarse discretization.

Many details were swept under the rug in the preceding description. A fuller version of the story can be found in the following sources, which represent just a few of the many papers on structure-preserving methods in linear algebra. A general survey of quadratic eigenvalue problems, including conventional linearizations, is given in [5]. A first attempt at structure-preserving linearizations for matrix polynomials is described in [4]. A forthcoming paper [2] introduces vector spaces of linearizations for matrix polynomials, and another presents linearizations for palindromic polynomials [3]. The work on the SFE project is the subject of a master's thesis [1].

References

[1] A. Hilliges, Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendungen in der Schienendynamik, Master's Thesis, Technical University Berlin, Germany, July 2004.

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[3] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Palindromic polynomial eigenvalue problems: Good vibrations from good linearizations*, in preparation.

[4] V. Mehrmann and D. Watkins, *Polynomial eigenvalue problems with Hamiltonian structure*, Electr. Trans. Num. Anal., 13 (2002), 106–113.

[5] F. Tisseur and K. Meerbergen, *A survey of the quadratic eigenvalue problem*, SIAM Rev., 43 (2001), 234–286.

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‡ Strictly speaking, it is a linearization only if –1 is not an eigenvalue; if –1 is an eigenvalue, the eigenvalue must be removed (deflated) first.