from *SIAM News*, Volume 36, Number 5, June 2003

Gamma: A Biography

GAMMA: Exploring Euler's Constant. *By Julian Havil, Princeton University Press, Princeton and Oxford, 2003, xxii + 268 pages, $29.95.*

BOOK REVIEW

By James Case

Like Martha Stewart, publishers know a good thing when they see it. A current example is the proliferation of biographies of important numbers. Eli Maor's history of e and Paul Nahin's story of $i = \sqrt{-1}$ have been joined by books about zero, π, and the Golden Mean. Now, from Princeton, comes a book on γ, the Euler–Mascheroni constant. Can histories of unity, $\sqrt{2}$, and the Catalan constant be far behind? How about Chaitin's maximally noncomputable Ω? Will success breed success until sequels and prequels proliferate like Rocky movies, or mushrooms after a rain?

Julian Havil is a master (read teacher) at Winchester College (read private secondary school) in England. He has taught there for thirty years, since completing his PhD at Oxford. Winchester College has traditionally been regarded as the best of the British secondary schools for mathematics. G.H. Hardy attended during the 1890s, as did Freeman Dyson half a century later. Declaring in the foreword to Havil's book that his own love of mathematics was born there, Dyson goes on to describe the book as "inspiring," and designed to "show students how enchanting mathematics can be," by "teaching the practical skills students need" in the context of the time "when these skills were first developed."

The time chosen is—appropriately, in Dyson's opinion—the 18th century. Presided over by the genius of Leonhard Euler, it was the age that "created the language and the style in which mathematics has developed ever since." It happened because "the tricks and ideas of higher mathematics arose naturally out of the problems of the day." Dyson deems Euler's ideas "simple enough to be accessible, and deep enough to give a feeling for the beauty of real mathematics." The book, he goes on to say, is "centered on Euler's personality and the ideas he left for his successors to use and ponder." This alone should make it an interesting read.

The book is not intended for professional mathematicians. Anyone with a command of univariate calculus can read it, and should surely consider doing so if seeking an undemanding introduction to higher mathematics. According to the author's introduction, the first 11 chapters deal with theory and the last five with applications. Many (though not all) of the latter concern analytic number theory. The book ends with 40 pages of rather terse appendices, the longest of which explain the rudiments of complex variables, and the process by which Rie-mann extended the zeta function to the entire complex plane. It is remarkable how much 18th-century mathematicians were able to learn about the distribution of primes with so shaky a command of complex analysis.

The Euler–Mascheroni constant γ was first identified by Euler in or about 1735. Mascheroni was an Italian geometer best remembered for proving that any geometric construction that can be carried out with ruler and compass can in fact be carried out with compass alone. He never had much to say about

$$\gamma = \lim_n [H_n - \ln(n + \alpha)], \tag{1}$$

where ln denotes the logarithm to the base e, and H_n is the nth partial sum of the harmonic series $\sum_n 1/n$. So it is only natural that the book's first two chapters should deal with Napier's development of logarithms and the harmonic series, respectively. The foregoing definition of γ is unambiguous because the indicated limit is altogether independent of $\alpha > -n$. Many authors set α equal to zero, but strong numerical evidence suggests that the sequence converges most rapidly when α is roughly 1/2. The chapter on logarithms describes ancient attempts to reduce multiplication, division, and root extraction to addition, as well as Napier's method for accomplishing the feat and his contemporaries' assessments of the method. His accomplishments were particularly appreciated by the astronomers of the day, one of whom reckoned that the use of logarithms at least doubled his productivity. The chapter on the harmonic series, and the following one on subseries of it, lie farther from the beaten track.

Havil identifies three "unexpected" properties of the sequence $\{H_n\}$: (i) it diverges; (ii) it does so without assuming any integer values greater than 1; and (iii) except for $H_1 = 1$, $H_2 = 1.5$, and $H_6 = 2.45$, the decimal expansion of H_n is always a nonterminating decimal, because the denominator of any higher H_n—when expressed as a single improper fraction—always involves a prime factor greater than 5. This Havil proves by induction on n. He also offers several different demonstrations of (i) and establishes a stronger form of (ii) according to which there are no integers among the differences $H_n - H_m$ for which $1 \leq m < n$. His chapter on subseries of the full harmonic series mentions the ten (convergent) Kempner series obtained by deleting all the terms $1/n$ in which n contains a particular decimal digit (such as 7), as well as the more important (divergent) series $\sum 1/p$ of prime reciprocals. The proof that the limit defining γ actually exists is deferred until the ninth chapter, by which time the reader has already met with the real zeta function $\zeta(x) = \sum_n 1/n^x$, along with some of Euler's early results concerning it.

Havil quite reasonably dates the birth of analytic number theory to Euler's enunciation of the identity

$$\zeta(x) = \sum_n 1/n^x = \Pi_p 1/(1 - p^{-x}), \tag{2}$$

in which $x > 1$ and the product extends over all primes p. This he proves in more than usual detail, and exploits in small ways. He then combines it with the principle of inclusion and exclusion to prove that the probability that an arbitrarily chosen pair of positive integers will have no common divisor greater than 1 is $1/\int_0^\infty u\,du/(e^u - 1) = 6/\pi^2$. And while one may wonder what the target teen-aged reader will make of such a proof, one can confidently expect that, while only a few will persevere to the end, the many who fall by the wayside will still profit from the experience. His proof makes use of familiar facts about the gamma function, which he describes at some length, and relates to γ in various ways.

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Having established that the limit of the sequence $\{H_n - \ln(n)\}$ actually exists, and is approximately equal to 0.577 215 664 901 532 860 606 5. . . , Havil explores its expansions in both decimals and partial fractions. He also points out that no one yet knows whether γ is rational, irrational, or transcendental, and recounts Hardy's famous offer to relinquish the Savilian chair of mathematics at Cambridge—once occupied by Newton himself—to anyone who could convince him of its rationality or lack thereof. He also mentions that γ turns up in countless sums, products, and integrals, of which we mention only

$$\gamma = -\int \exp(-u) \ln(u)\, du = -\int \ln \ln(1/x) dx = \lim_n (n - \Gamma(1/n)), \tag{3}$$

where the first integral extends over the positive reals, while the second goes only from 0 to 1. It also turns up in oft used expansions of the cosine integral $Ci(x)$, exponential integral $Ei(x)$, and logarithmic integral $Li(x)$.

Havil's favorite applications of 18th-century mathematics have to do with the lore of prime numbers. Nevertheless, he devotes a chapter each to nonstandard applications of logarithms and the harmonic series. The chapter titled "It's a Logarithmic World" begins with a discussion of Shannon information, followed by something called Benford's law, "which purports to explain the oft observed fact that tables of logarithms wear unevenly, the first few pages sustaining most of the damage," and ends with an explanation of the fact that the partial quotients $\{a_n\}$ in the partial fraction expansion $x = a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + \ldots)))$ of a randomly chosen $x \in (0,1)$ occur with the following frequencies, expressed as percentages:

k	1	2	3	4	5	6	7	8	9+
Prob($a_n = k$)	41	17	9	6	4	3	2	2	16

(4)

In contrast, the chapter titled "It's a Harmonic World" contains discussions of (among other things) card shuffling, the QUICKSORT algorithm, and the destructive testing of beams.

That chapter also offers an explanation for the fact (hypothesis?) that the expected number of years of record rainfall in a city in which rainfall has been recorded for n consecutive years is H_n. Thus, in New York City, where records have been kept for 160 years, the expected number of record years is $H_{160} = 5.65$, while in Oxford, UK—where records have been kept for 234 years—the corresponding expectation is $H_{234} = 6.03$. The relevant records indicate that Oxford has in fact enjoyed but five years of record rainfall during the last 234 years, while New York has witnessed six. It would be difficult indeed to reject the foregoing hypothesis on the basis of such data! A particularly timely implication of the surprisingly small values of H_n (for example, H_{1000} and $H_{1,000,000}$ are just 7.49 and 14.39, respectively) is that, *without climatic change*, record years would be very rare even over long time spans.

Havil's real target applications have to do with the distribution $\pi(x)$ of the prime numbers, and he devotes the final two chapters of the book to such applications. Chapter 15 employs elementary methods, while Chapter 16 explores Riemann's contributions to the subject. Although little is proven in either chapter, Havil makes good use of computer studies, and manages to convey the excitement and wonder that have for so long surrounded his subject. It is tempting to speculate that the Riemann hypothesis will indeed be settled during the new century, and that the eventual solver will owe his or her original inspiration to such a book as this.

James Case writes from Baltimore, Maryland.