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Does Marilyn Know her Game Theory?

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The following question appeared in the "Ask Marilyn" column (Marilyn vos Savant is listed in the *Guinness Book of World Records Hall of Fame* for the highest IQ) in *Parade Magazine* on March 31, 2002:

"Say you're in a public library, and a beautiful stranger strikes up a conversation with you. She says: 'Let's show pennies to each other, either heads or tails. If we both show heads, I pay you \$3. If we both show tails, I pay you \$1. If they don't match, you pay me \$2.' At this point, she is shushed. You think: 'With both heads 1/4 of the time, I get \$3. And with both tails 1/4 of the time, I get \$1. So 1/2 of the time, I get \$4. And with no matches 1/2 of the time, she gets \$4. So it's a fair game.' As the game is quiet, you can play in the library. But should you? Should she?—*Edward Spellman, Cheshire, Connecticut.*

Marilyn's answer appeared the following week, in her column of April 7:

"The woman in the library said: 'Let's show pennies to each other, either heads or tails. If we both show heads, I pay you \$3. If we both show tails, I pay you \$1. And if they don't match, you pay me \$2.' Should you play? No. She can win easily. One way: If she shows you twice as many tails as heads, she wins an average of \$1 for every six plays."

Marilyn's answer, which has the stranger playing twice as many tails as heads and winning \$1 in six plays, is wrong if the opponent elects to play a pure strategy of tails *all* the time! In that case, the stranger and her opponent break even. Given the beautiful stranger's strategy of playing twice as many tails as heads, what she wins depends on her opponent's strategy. If the opponent plays heads and tails randomly, 50–50, the stranger will win \$1 for every six plays, as Marilyn stated. Nothing in the problem as stated, however, would stop the opponent from playing tails *all* the time. In this case, they would break even (the stranger's winnings, on average, would be $2 + 2 - 1 - 1 - 1 - 1 = 0$).

Basic Game Theory Analysis

In contrast, if she randomly plays three heads and five tails, it can be shown, based on game theory, that no matter what strategy her opponent uses, she will win, on average, \$1 every eight games. Based on basic game theory, we now explain why this is the optimal strategy for the beautiful stranger.

The problem posed in the "Ask Marilyn" column is a two-person zero-sum game, i.e., the winnings of the two players must sum to zero. In other words, what one player wins the other loses. The winnings for the opponent of the beautiful stranger are shown in the table.

	Beautiful Stranger's Strategy	
	Heads	Tails
Opponent's Strategy		
Heads	3	–2
Tails	–2	1

Because the maximum of the row minimum values (–2) is not equal to the minimum of the column maximum values (1), this game does not have a saddle point. In the absence of a saddle point, the optimal strategies for the beautiful stranger and her opponent will necessarily be mixed randomized strategies.

Let x be the probability that the opponent plays heads, and $1 - x$ the probability that the opponent plays tails. Analogously, the beautiful stranger plays heads with probability y and tails with probability $1 - y$. The expected rewards for the opponent are then $3x + (-2)(1 - x)$ against heads, and $-2x + (1)(1 - x)$ against tails. The opponent's optimal strategy is given when these two expected rewards are equal, i.e., when

$$3x + (-2)(1 - x) = -2x + (1)(1 - x).$$

Solving this equation gives the strategy (3/8, 5/8). In other words, on average, the opponent should play three heads and five tails. By following this strategy, the opponent would *lose* a minimum of 1/8 of a dollar per game. Analogously, the optimal strategy for the beautiful stranger is obtained by solving the following equation for y:

$$3y + (-2)(1 - y) = -2y + (1)(1 - y).$$

The optimal strategy for the beautiful stranger is (3/8, 5/8). She, too, should play, on average, three heads and five tails, for an average winning per game of 1/8 of a dollar.

If the beautiful stranger plays her optimal strategy of (3/8, 5/8) and her opponent plays pure heads, the opponent will win $(3/8)(3) + (5/8)(-2) = -1/8$ dollar per game and the beautiful stranger will win 1/8 dollar per game. Similarly, if the beautiful stranger plays her optimal strategy of (3/8, 5/8) and her opponent plays pure tails, the opponent will win (3/8)(–2) + (5/8)(1) = –1/8 dollar per game and the beautiful stranger will win 1/8 dollar per game. Given that, if the beautiful stranger plays her optimal strategy of (3/8,5/8) and her opponent plays either a pure strategy of heads or a pure strategy of tails, the beautiful stranger will win, on average, 1/8 of a dollar per game. In fact, any linear combination of these pure strategies on the part of her opponent will still result in the beautiful stranger winning 1/8 of a dollar per game. Hence, if the beautiful stranger plays, on average, three heads and five tails, then *no matter what strategy* her opponent plays, she will win an average of 1/8 of a dollar per game, or $1 in eight games.

Marilyn's suggested strategy of (1/3, 2/3) for the beautiful stranger is not optimal and makes the outcome of the game dependent on the strategy chosen by her opponent. Consider the following pairs of pure strategies for the opponent and resulting outcomes for the beautiful stranger if she follows Marilyn's recommended strategy of (1/3, 2/3):

- Opponent plays heads always, beautiful stranger wins (1/3)(–3) + (2/3)(2) = 1/3 dollar per game.
- Opponent plays tails always, beautiful stranger wins (1/3)(2) + (2/3)(–1) = 0 dollar per game.
- Finally, if the opponent randomly plays heads and tails, (1/2,1/2), the beautiful stranger will win an average of $1 for every six plays, as stated by Marilyn.

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