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Disturbing Theorems and the Quest for Ideal Election Mappings

Chaotic Elections: A Mathematician Looks at Voting. *By Donald G. Saari, American Mathematical Society, Providence, Rhode Island, 2001, 159 + xiii pages, $23.00.*

BOOK REVIEW

By James Case

Scientific interest in the seemingly paradoxical outcomes to which the election process can lead first blossomed during the late 18th century, among members of the French Academy of Sciences. Borda, Condorcet, and Laplace, among others, identified important paradoxes and proposed ways of avoiding them. They also left many unanswered questions, the bulk of which remained open during the 1970s, when Don Saari first became aware of them. The earlier investigators—few of whose contemporaries took democracy at all seriously—recognized from the start that "the popular will" is a distressingly nebulous concept, potentially sensitive to the methods (algorithms?) used to determine it. They also realized that "majority vote" leaves much to be desired whenever an election is contested by more than two candidates and suggested alternative voting schemes. The fact that they raised more issues than they were able to resolve does not diminish the magnitude of their achievements.

Attention to the subject was relatively rare—and largely unproductive—during the 19th and early 20th centuries. The drought finally ended in 1951, with economist Kenneth Arrow's book *Social Choice and Individual Values,* which contains what quickly became known as "Arrow's impossibility theorem." Arrow's theorem appears to foreclose—in the manner of Gödel's incompleteness theorem—once-promising avenues of research, and was cited by the relevant Swedish authorities in 1972, when Arrow shared the Nobel Memorial Prize for Economic Science with (Sir) John R. Hicks.

Arrow's "Impossibility Theorem"

Arrow's approach to the study of elections postulates that each voter could, if pressed, rank the entire slate of candidates in order of preference. He supposed these individual preferences to be "complete," in the sense that each candidate is ranked, "strict," in the sense that no voter is indifferent between any pair of candidates, and "transitive," in the sense that if A is preferred to B, and B to C, then A is preferred to C. Arrow deemed all this appropriate despite the fact that the most common election formats—in which voters are asked to vote for at most one, two, or three candidates—extract only a fraction of this information about individual preferences. The purpose of an election, in Arrow's formalization, is to construct a consensus (or "societal") ordering as consistent as possible with the voters' individual preferences. In the standard "vote for k" formats, the consensus ordering merely ranks the candidates according to the number of votes received by each. Arrow required consensus orderings to be complete and transitive, though not—since ties are always possible—strict.

A "preference profile" in an election between three candidates—call them Bush (B), Gore (G), and Nader (N)—is a six-component vector, the coordinates of which specify the number of voters whose preference orderings are respectively *BGN*, *BNG*, *GBN*, *GNB*, *NBG*, *NGB*. A preference profile in an N-candidate election is a (similarly constructed) $N!$-component vector. Preference profiles constitute an effective "data compression" device for elections in which millions of votes are cast for a relative handful of candidates.

Arrow's theorem concerns "election mappings" $F{:}D \rightarrow R$ whose domains D consist of preference profiles and whose ranges R contain nonstrict (consensus) orderings. The theorem asserts that the mapping F that satisfies two seemingly natural requirements is unique. One need not know the nature of those requirements to understand the impact Arrow's result has had on the field. It is described as an impossibility theorem because the mapping that satisfies both of the requirements Arrow saw fit to impose represents a manifestly undemocratic method of ordering candidates. This has led many to abandon the study of the election process on the ground that socially acceptable consensus orderings—like methods of constructing them—fail to exist.

Saari deems such judgments premature since society accepts (at least tacitly) a particular consensus ordering, along with the method by which it was constructed, every time it chooses to abide by the results of an election. The fact that the most common election formats fail to comply with Arrow's (seemingly innocuous) requirements does not deter society from accepting either the results or the methods used to obtain them. This is true despite the fact that the more common formats are demonstrably less revealing of the public will than others less familiar. To Saari, that seems reason enough to continue (indeed encourage) the study of consensus orderings and methods for obtaining them.

The unique mapping F that satisfies both of Arrow's requirements is known as the "dictator map" because it allows a particular voter to "dictate" society's preferences by mapping every profile $p \in D$ onto that element of R that represents his (the dictator's) individual preference ordering, whether or not any of the other voters concur in whole or in part. How could this be described as a "consensus" ordering of the candidates, and how could so undemocratic a process satisfy requirements intended to identify means of obtaining such preference orderings?

The answers to such questions depend on careful examination of Arrow's requirements. The first, known as the Pareto, or "unanimity," requirement, states only that if every voter prefers candidate A to candidate B, then society as a whole should prefer

A to B as well. Saari has no quarrel at all with that. The second requirement, known as the "independence of irrelevant alternatives," states that if two profiles p and p' in D are such that each voter ranks A in the same way relative to B in each, then the elements $F(p)$ and $F(p')$ of R to which an election mapping F sends the profiles p and p' should also rank A in the same way relative to B. That, according to Saari, is an unreasonable requirement. His objections are plainly illustrated by the Y2K presidential vote in Florida, which saw George Bush (B), Al Gore (G), and Ralph Nader (N) garner the most votes.

Suppose that 501,000 Florida voters had ranked the candidates in the order *BNG*, while 499,000 placed them in the order *GNB*, 5000 in the order *NGB*, and none in any of the other three orders. Then $p = (0, 501, 0, 499, 0, 5)$, in thousands of voters. Next, suppose that 3000 of the voters revise their ordering from *NGB* to *GNB*, so that $p' = (0, 501, 0, 502, 0, 2)$, again in thousands of voters. In either case, 501,000 voters prefer B to G, while 504,000 voters prefer G to B, so that every voter ranks G in the same way relative to B under both p and p'. Yet the election mapping F that counts only first-place votes yields $F(p) = BGN \neq GBN = F(p')$, thereby violating (because they rank B and G differently) the requirement that election outcomes be independent of irrelevant alternatives.

The imposition of that appealing yet exclusive requirement on election mappings led Arrow to the conclusion that "ideal" election mappings don't exist. Yet his results shed little light on the nature of "best possible" election maps, a goal toward which Saari and others have made notable progress.

Disturbing Theorems from Saari

Saari begins his search by classifying the more common means of evaluating election mappings. The most common by far is the so-called *plurality vote*, according to which each voter casts a single vote for a single candidate, and the candidate with the highest total wins. Plurality voting makes no attempt to discover which candidates rank second or third in public esteem. Another common scheme instructs voters to vote for k ($< N$) candidates, without placing them in any particular order. City council members (among others, including SIAM board and council members) are often elected by this method. If voters are instructed to vote for all but one candidate, the result is equivalent to *antiplurality voting,* in which voters vote against one candidate. This method is sometimes employed iteratively, to eliminate candidates one by one.

All three of the foregoing methods are examples of what Saari describes as *positional methods*, whereby each voter assigns $w_1 \geq w_2 \geq \ldots \geq w_N$ points to her first-choice, second-choice, . . . , and last-choice candidate. Perhaps the most famous positional method is the so-called *Borda Count*, whereby each voter assigns $N - j$ points to her jth-choice candidate. This method, first proposed by Jean Charles de Borda, a.k.a. Count Borda, in 1770, caused his contemporaries to wonder what was so special about that particular choice of weights. Only in recent years has Saari discovered a reasonably complete answer to their question.

A method that does not belong to the class of positional methods is called *approval voting*, whereby each voter votes for the subset of the candidates of which he or she approves. It is not a positional method because the total number of votes cast—or total weight assigned—by a given voter is unspecified. Another nonpositional method of voting assigns each voter a given number of points, and allows him or her to distribute them at will among the candidates.

Saari found positional methods to be fruitful objects of study, both because they include a number of the most commonly employed voting schemes, and because he was able to prove a number of disturbing theorems about them. Among the most damning is his demonstration that, with $N > 2$ candidates, it is possible to construct a preference profile p for which candidate 1 wins when voters are instructed to vote for one candidate, candidate 2 wins when voters are instructed to vote for two candidates, . . . , candidate $(N - 1)$ wins when voters are instructed to vote for $N - 1$ candidates, and candidate N wins when the Borda Count is used to determine the consensus ranking. In other words, the scoring method may have as much to do with determining the outcome of a particular election as does the profile of individual preferences. Saari displays such profiles for $N = 3$ and $N = 4$ before pointing out that there exist additional profiles p for which $N! - (N - 1)!$ of the possible $N!$ orderings of the candidates emerge as consensus orderings when the scoring method is allowed to vary over the entire range of positional methods.

A particularly alarming result concerns the outcomes of committee meetings in which pairs of candidates are compared in sequence, the loser to be eliminated at each stage. Saari asks the reader to imagine a department meeting at which a new calculus text is to be chosen from eight candidates, A, B, . . . , H. Ten members of the 30-member department rank the candidates in each of the orders *ABCDEFGH*, *BCDEFGHA*, and *CDEFGHAB*. Candidate H is obviously the least highly regarded, since nobody ranks it higher than sixth, while an equal number rank it at the very bottom. Yet if the first contest is between F and G, while each of the remaining six is matched against the winner of the most recent contest in the order E, D, C, B, A, H, it will be seen that H emerges as the ultimate winner in a sequence of unanimous or landslide (two-thirds plurality) two-candidate elections! Such is the power vested in the one who controls the agenda.

Where the Math Societies Stand

Much of the book is filled with examples of the ways in which a seemingly democratic election process can result in an outcome (consensus ordering) that obviously distorts the popular will. Indeed, Saari—who is frequently asked to speak on the subject—often jestingly offers to rig an organization's next election by sampling members' preferences and designing a "democratic voting method" that involves all candidates and is guaranteed to produce a predetermined result. Theorems, though often stated, are seldom proven. Citations are provided, however, usually to previous publications of Saari's.

Saari is particularly concerned with the ability to manipulate multicandidate elections by strategic voting of the sort engaged in by the Nader supporters in Florida who voted for Gore in an effort to defeat Bush, or by voters who "single shoot" for a candidate in an election in which they are entitled to vote for several. His conclusions on the matter are well summarized in the text of his

Theorem 9, which reads in part: "The procedures which are most susceptible to a successful manipulation are the plurality and anti-plurality votes. The unique procedure which is least susceptible to a successful manipulation is the Borda Count." Although the theorem in question applies only to three-candidate elections in which a small fraction of the electorate plans to vote strategically, Saari seems to feel that similar conclusions hold in most other circumstances.

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It seems ironic, given the obvious susceptibility of the election process to mathematical analysis—along with the several unique properties of the Borda Count—that the method is not in use by the American Mathematical Society, the Mathematical Association of America, or SIAM. Has any of the three even bothered to form a committee to evaluate the pros and cons of adopting it? Might it actually be, as Saari seems to think, a better way to extract a consensus from the usual welter of conflicting individual preferences? And how, if mathematicians themselves are hesitant to act on the results of mathematical analysis, can they expect others to be less so?

Any responsible evaluation committee would have to look long and hard at—among other things—Saari's resolution of the Condorcet paradox, whereby the Borda Count might fail to elect the one candidate in a field of three who is capable of defeating either of the other two in head-to-head competition. Whether or not the organization elected to adopt the method, the committee's published report—evaluating the strengths and weaknesses of the method—could easily become the most influential document ever published on the subject of voting procedures. The academic community would subject its conclusions to intense scrutiny, while societies and jurisdictions contemplating revision of their own election procedures would feel obliged to consult it.

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