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A Welcome Addition to the Chaos Literature

Chaotic Transitions in Deterministic and Stochastic Systems: Applications of Melnikov Processes in Engineering, Physics, and Neuroscience. *By Emil Simiu, Princeton University Press, Princeton and Oxford, 2002, xiv+224 pages, $49.50.*

BOOK REVIEW

By James Case

For all that has been written about chaos, there seems to be a gap between the popular expositions (of which James Gleick's *Chaos* is deservedly the best known) and the scholarly treatises. There ought to be an intermediate text that quickly conveys the reader from a standing start to at least a few of the more accessible applications. Now Emil Simiu, a fellow at the National Institute of Standards and Technology and a research professor in the Whiting School of Engineering at Johns Hopkins University, has written such a book. One can imagine others, stressing different aspects of the theory and aimed at different areas of application, but here at least is one.

Among Simiu's applications are small boats at anchor, which are kept in virtually perpetual motion by the action of wind and waves. If wind effects are negligible, and if successive wave fronts are parallel to the line from stem to stern, such a boat will neither pitch nor yaw. It will merely roll from side to side, in a motion governed by a single second-order system of ordinary differential equations. The boat will continue to roll to and fro until θ—the angle between the boat's plane of symmetry π and a vertical plane that meets π in a horizontal line—exceeds a critical value θ_c, usually between 30° and 35°, at which time it will capsize.

Reduced to nondimensional form, the equations of rolling motion become

$$\dot{x} = y$$
$$\dot{y} = x + ax^3 - \varepsilon[by + cy|y| - f(t)], \qquad (1)$$

where $x = \theta$, $t = \omega\tau$, a, b, c, and ω are measured constants, and overdots denote differentiation with respect to the dilated clock time t. The unperturbed system is of the Hamiltonian form $\ddot{x} = -V'(x)$, where $' = d/dx$, $H = y^2/2 + V(x)$, $V(x) = x^2/2 + ax^4/4$, and the point $x = y = 0$ lies at the bottom of a lens-shaped basin of attraction. Wave action appears only in the perturbed system, where it is incorporated into the forcing term $f(t)$. If the waves are sufficiently violent, the "state vector" $(x(t),y(t))$ can escape from its basin of attraction, causing the boat to capsize. Opinions vary as to whether $f(t)$ should vary harmonically, periodically, almost periodically, stochastically, chaotically, or whatever.

In addition to boats at anchor, a large number of physical devices can be modeled as second-order Hamiltonian systems, perturbed by noise. Such systems can exhibit arbitrarily many basins of attraction, and their state vectors can be driven by noise to jump repeatedly from one such basin to another. If the noise is loud enough, the resulting motion is all but certain to be chaotic. Emil Simiu illustrates the use of specific techniques, mainly those of Melnikov, for analyzing the motion of such systems. Although the techniques have analogues in higher dimensions, the underlying ideas are most easily explained in just two. Moreover, the relative abundance of significant applications in the plane suggests that a two-dimensional introduction to the subject, easily accessible to engineers and scientists, should be of considerable value.

Most of the systems discussed in the book are of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{\mu}) + \varepsilon \boldsymbol{g}(\boldsymbol{x},t;\boldsymbol{\mu}) \qquad (2)$$

where $(\boldsymbol{x} = x_1,x_2)^T$, $\boldsymbol{f} = (f_1,f_2)^T = (H_y,-H_x)^T$, and $\boldsymbol{g} = (g_1,g_2)^T$, while $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is a vector of parameters. When H is of the form $y^2/2 + V(x)$, x and y can be referred to as the position and momentum components, respectively, of the state vector $\boldsymbol{x}$. Potential wells are features of potential functions $V(x)$, and typically correspond to basins of attraction in the plane containing all possible state vectors $\boldsymbol{x} = (x,y)^T$. On occasion, Simiu also considers three-dimensional systems—in which $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and $x_3 = z$ is a slowly varying quantity such that $\dot{z} = \varepsilon g_3(x, y, z, t; \boldsymbol{\mu})$. In the absence of specific warnings to the contrary, the func-tions H, V, g_1, g_2, and g_3 are assumed to be many times differentiable in all arguments.

The basins of attraction of such systems can be as distinct as those in Figure 1, or as intertwined as those in Figure 2. In three dimensions, the geometry can be vastly more complex. In all cases, it is important to know whether and how often the state vector of the perturbed system can be expected to jump back and forth between the several basins. The Melnikov method is designed to answer such questions. When jumps are possible, the position component of the state vector tends to oscillate first around one value, and then around another, as indicated in Figures 3a and 3b, obtained by forcing the system of Figure 1 in different ways. Can you guess which history corresponds to harmonic, and which to stochastic, forcing? (Answer: (a) is harmonic.)

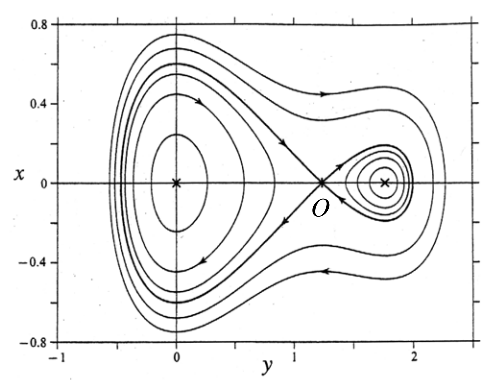

Figure 1. *Homoclinic orbits and typical periodic orbits in an unperturbed system (from Allen, Samelson, and Newberger, 1991). All figures from* Chaotic Transitions in Deterministic and Stochastic Dynamical Systems.

Saddlepoints like the ones at the points O in Figures 1 and 2 are of particular significance in phase plane analysis. They are isolated stagnation points (in the sense that $\boldsymbol{x}$ vanishes nowhere else in a neighborhood of O) at which two stable solutions end and two unstable ones begin. The Melnikov function $M(\cdot)$ assumes its simplest form in cases in which at least one of the unstable solutions beginning at a saddlepoint O bends back on itself to become a stable solution, ending where it began. This occurs on both sides of Figures 1 and 2. In such cases, there are many solutions $\phi(t)$ of the unperturbed system (2) for which $\phi(\infty) = \phi(-\infty) = O$. If P is a point on an arc Γ traversed by a continuum of such solutions, and if $\phi_p(\cdot)$ denotes the particular solution for which $\phi(O) = P$, then every point Q of Γ is characterized by the time $t = t_Q$ at which $\phi_p(t) = Q$. Indeed, $\phi_p(t + t_Q) = \phi_Q(t)$ is the solution of (2) for which $\phi(O) = Q$. The Melnikov function is then

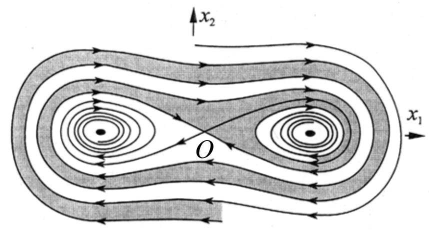

Figure 2. *Point attractors and their basins of attraction for an unforced, dissipative double-well oscillator (after Thompson and Stewart, 1986).*

$$M(t_Q) = \int [\quad \boldsymbol{f}(\phi_p(u)) \times \boldsymbol{g}(\phi_p(u),\ u + t_Q)]du, \tag{3}$$

the integral (3) running from $-\infty$ to ∞, and the cross product of $\boldsymbol{a} = (a_1, a_2)^T$ and $\boldsymbol{b} = (b_1, b_2)^T$ being the scalar $\boldsymbol{a} \times \boldsymbol{b} = a_1 b_2 - a_2 b_1$. $M(\cdot)$ contains a surprising amount of qualitative information about the solutions of (2), which correspond to small values of $\varepsilon > 0$.

As ε increases from 0, a particular Γ typically bifurcates into distinct arcs Γ' and Γ'', traversed, respectively, by stable and unstable solutions of (2). For small values of ε, both Γ' and Γ'' will approximate Γ. Hence, a perpendicular to Γ through a point $Q \in \Gamma$ will meet Γ' at Q' and Γ'' at Q''. Let $A(t_Q;\varepsilon)$ denote the distance between Q' and Q''. Direct calculation reveals that

$$\Delta(t_Q;\varepsilon) = \varepsilon M(t_Q) + O(\varepsilon^2) \tag{4}$$

and hence that, for sufficiently small values of ε, Γ' and Γ'' cross near simple zeros of $M(\cdot)$. A significant portion of the second chapter is devoted to calculating $M(\cdot)$ for specific types of perturbation $\boldsymbol{g}$. The integral (3) is shown, in many cases of interest, to differ by at most a constant from a convolution. In other words, $M(t_Q) = h * f - k$, where $f = f(t)$ is a time-dependent forcing function and $h = h(t)$ is an impulse response. The utility of the Melnikov function is limited by the fact that, since the integral in (3) extends from $-\infty$ to $+\infty$, the relationship between $f(\cdot)$ and $M(\cdot)$ is anticipative rather than causal.

The principal results concerning $M(\cdot)$ are that the "separatrices" Γ' and Γ'' must cross near simple zeros thereof, and that—as suggested by Figure 3—the motion of the system is chaotic whenever the forcing term $f(t)$ is large enough to cause jumping from one basin of attraction to another. When $f(\cdot)$ becomes a stochastic process, $M(\cdot)$ becomes one too, and the integral in (3) becomes stochastic. Yet the resulting "Melnikov process" retains much of its former significance, yielding an estimate of the mean escape time from a given basin of attraction, of the probability of nonescape during a specified time interval, and more. Many of the stated results remain valid even when $f(t)$ is non-Gaussian.

To make the book reasonably self-contained, Simiu is obliged to include a chapter on deterministic chaos, including Lyapunov exponents, Cantor sets, fractal dimensions, Smale's horseshoe map, the shift map, and symbolic dynamics, along with a chapter on stochastic processes, before proceeding to stochastic chaos. The more technical aspects of all this are relegated to seven appendices.

Readers already acquainted with chaos and stochastic processes will be able to skip more or less directly to the second part of the book, on applications. Included here are chapters on boat capsizing, wind-induced ocean currents, auditory nerve fibers (which can also experience abrupt transformations in state), sudden transitions of a heavily loaded and transversely excited column from one deformed state to another, and more.

For this reviewer, the chapter on "stochastic resonance" was particularly enlightening. When certain systems of the type described by equation (2) are excited by both noise and a periodic signal, it is possible to increase the signal-to-noise ratio by the paradoxical means of increasing the noise intensity. This phenomenon is known as stochastic resonance. The Melnikov approach provides a unifying approach to such problems, permitting comparison between this and other (typically more mundane) means of improving the signal-to-noise ratio. The approach reveals, in particular, that the improvement is greatest when the spectral power of the noise is concentrated at or near the frequency at which a readily computable quantity known as the "Melnikov scale factor" achieves its maximum.

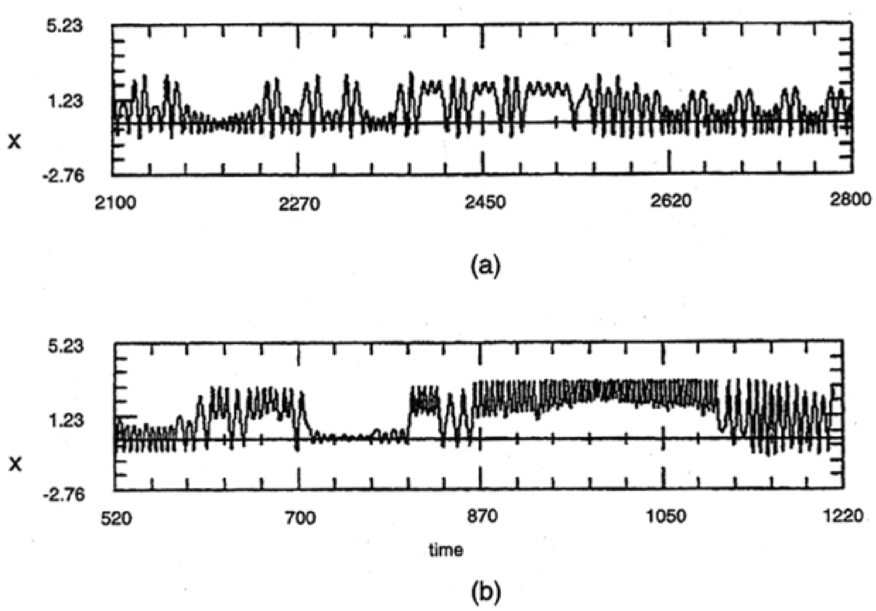

Figure 3. *Records of chaotic motion for (a) harmonic forcing and (b) realization of stochastic forcing.*

All in all, Simiu has written an interesting and broadly accessible book—a welcome addition to the rapidly expanding literature of chaos.

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