# Understanding Tides—From Ancient Beliefs to Present-day Solutions to the Laplace Equations

**Tides: A Scientific History.** *By David E. Cartwright, Cambridge University Press, Cambridge, UK, 1999, 272 pages, $69.95.*

David Edgar Cartwright spent more than forty years at the forefront of oceanographic research, working mainly at the UK Institute of Oceanographic Sciences, the Scripps Institute of Oceanography, and the NASA–Goddard Space Flight Center. Upon retirement, he undertook the task of writing a comprehensive history of "tidal science." His timing could hardly have been better, since modern computers, classical hydrodynamics, and satellite technology had recently combined (during the 1980s and 1990s) to answer all or most of the traditionally asked questions about tidal motion. A definite milestone has been passed—one toward

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**BOOK REVIEW**

*By James Case*

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which the likes of Bacon, Galileo, Descartes, Kepler, Daniel Bernoulli, Euler, Colin Maclaurin, Laplace, Thomas Young, George B. Airy, Kelvin, George Darwin, and Horace Lamb directed their attention. A unified account of their efforts, frustrations, and achievements in this particularly practical branch of earth science seems long overdue.

In the opening chapters, Cartwright dutifully catalogs the facts of tidal motion, as ancient, medieval, and Renaissance scholars knew them: High and low tides occur on a semi-diurnal (twice-daily) basis and differ more in some places than in others; high and low tides in a given port are respectively higher and lower on some occasions than others; "spring" (the highest of several consecutive high) and "neap" (the lowest of several consecutive low) tides occur at intervals of roughly 15 (lunar) days; spring tides tend, on average, to be highest near an equinox.

The most famous of the ancient authorities—including Aristotle—are notably silent on the subject of tidal motion. Most of them lived on the Mediterranean, where tides are insignificant. The venerable Bede (672–735), of Jarrow Abbey, Northumbria, noted that the hour of high water marches southward, at a relatively constant rate, along the east coast of England. Francis Bacon (1561–1626) knew also that the hour of high water advances generally northward from Gibraltar to the North Sea, and recommended that tidal records be kept along the African coast and in the new world.

The first known tide-table for European waters is thought to be the work of John, Abbott of Wallingford, who died in 1213. It predicts the hours of "flod at london brigge" (high water at London Bridge), and was premised on the belief that high water will occur 48 minutes (4/5 of an hour) later tomorrow than today, some three hours later at London than at the mouth of the Thames. A document dated 1056 A.D. describes a similarly constructed table for the Chhien Thang river in China. What the producers of such tables could not do was explain why tides should occur in the first place. The thought occurred to both European and Oriental thinkers that the Earth itself might be alternately inhaling and exhaling sea water.

Cartwright is keenly aware that, despite the contributions of earlier scholars, his real story begins with Newton. Although he had relatively little to say about tidal motion, Newton did manage to get his successors off on the right track by observing that the net gravitational force (from the Earth and the moon) acting on a particle of fluid at the surface of a hypothetical canal encircling the Earth at the equator would be the same as that acting on a satellite orbiting immediately above the surface of said canal. A second net force (from the Earth and the sun) would have similar but weaker effects.

Newton also furnished a diagram from which the forces in question could easily be deduced, and he stated—without proof— that the effect of any one such force would be to deform the canal surface (or satellite orbit) from a circle into a (mildly eccentric) ellipse, the major axis of which passes through both the perturbing satellite and the gravitational center of the Earth. In so doing, he explained the semi-diurnal nature of tidal motion.

Whereas Newton's British successors showed little interest in tidal phenomena, several of his French admirers attempted to use his gravitational theory (as well as Descartes' rival "theory of vortices") to improve tidal prediction at ports along the Atlantic coast. But dissatisfaction with the results prompted the Académie Royale des Sciences to offer (in 1838) one of its prestigious prizes for the best philosophical essay on *le flux et le reflux de la mer*—the ebb and flow of the sea. Two years later, Daniel Bernoulli, Antoine Cavalleri, Euler, and Maclaurin shared the prize.

According to Cartwright, Cavalleri's essay bears the dubious distinction of being the last serious attempt in the history of science to justify Descartes' theory of vortices—with which French scholars were reluctant to part—by deducing Newton's inverse square law from it. Cavalleri, a Jesuit professor of mathematics at Cahors, also objected to Newton's theory of gravitation on the ground that it involved instantaneous action at a distance, a fact that continues to disturb more than a few modern scholars.

Maclaurin used Newton's theory of fluxions—on which he was a leading authority—to demonstrate that the shape of an otherwise spherical ocean of uniform depth in static equilibrium with the tidal force of a single deforming body of mass $m$ is a prolate spheroid whose major axis passes through that body. Moreover, the function describing the deformed surface can be expanded in ascending powers of $m$, the leading term being the ellipsoid asserted (without proof) by Newton. Maclaurin was also the first to discuss deflections resulting from the Earth's rotation, now known as Coriolis's effects.

Euler's main contribution was the realization that it is the horizontal component of the tidal force—as opposed to the vertical component, which is balanced by pressure on the sea bed—that determines tidal motion and causes the wavelike progression of high tide. But these were all minor contributions.

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The first major theoretical advance was made by Pierre Simon, marquis de Laplace, who formulated a system of three linear PDEs for the horizontal components $u(\theta, \phi, t)$ and $v(\theta, \phi, t)$ of ocean velocity relative to the Earth, and the vertical displacement $\zeta(\theta, \phi, t)$ of the ocean surface, where $\theta$ and $\phi$ represent the usual polar coordinates (co-latitude and east longitude) on the surface of the (not necessarily spherical) Earth. He found the equation of continuity to be

$$(\partial/\partial\theta)(vD\sin\theta) \; + \; (\partial/\partial\phi)(uD) \; + a \, \sin\theta \, (\partial\zeta/\partial t) \; = \; 0,$$

where $D \; = \; D(\theta, \phi)$ represents the depth of the ocean at position $(\theta, \phi)$ and $a$ represents a characteristic unit of length. Newton's laws then imply that

$$\partial u/\partial t \; - \; 2v\Omega\cos\theta \; = -(g/a) \, (\partial/\partial\theta)[\zeta \; - \; U \; - \; \delta U]$$

$$\partial v/\partial t \; - \; 2u\Omega\cos\theta \; = -(g/a \, \sin\theta)(\partial/\partial\phi)[\zeta \; - \; U \; - \; \delta U],$$

where $\Omega$ denotes the rate at which the Earth rotates about its axis, $U \; = \; U(\theta, \phi, t)$ is the scalar potential associated with the force field described by Newton, and $\delta U$ is a "secondary potential" introduced to account for the self-gravitation of the dynamically distributed ocean mass. Known to tidal scientists as the Laplace tidal equations, the equations remain the basis of tidal computation to this day. They generalize an earlier system of equations—developed by d'Alembert—that corresponds to the special case $\Omega \; = \; 0$. The coefficient $f \; = \; 2\Omega\cos\theta$ is now known as the Coriolis frequency, while the term Coriolis acceleration is routinely applied to the vector $(fv, -fu)$ by authors unaware that Maclaurin and Laplace had recognized its effect on tidal motion before Coriolis ever drew breath.

Laplace's main discovery concerning his tidal equations—in addition to the observation that the general case can be re-duced to the special case $\Omega = 0$—was that, because the equations are dynamically stable for realistic values of the physical constants involved, the frequencies of any and all solutions periodic in time must coincide with frequencies present in the forcing term $U \; + \; \delta U$. There are many such frequencies, due to various astronomical periodicities. Indeed, by 1939, when the Deutsche Hydrographische Institut at Hamburg constructed (for the presumed benefit of the U-boats then under construction) the mother of all (analog, single-purpose) tide-predicting machines, no fewer than 62 harmonic components of $U \; + \; \delta U$ had been identified.

From the fact that the trigonometric expansion of $U$ contained terms involving the cosines of $\Omega t$ and $2\Omega t$, as well as time-independent arguments, Laplace concluded that there must exist at least three "species" of tidal waves. The first, corresponding to terms independent of time, are long-period tides—typically of low amplitude—which would not be apparent to the casual dockside observer, but which careful measurements had already begun to detect. The second, corresponding to terms involving $\Omega t$, are the main cause of the long-noted differences between successive high- or low-water heights. And the third, corresponding to terms involving $2\Omega t$, cause the familiar semi-diurnal tides.

The data available to Laplace—indeed all tidal data predating the invention of automatic tidal recording devices[*]—identified only the heights and times of high (and possibly low) water. Only later did plots like that shown in Figure 1 become available. Recorded between September 6 and 21, 1831, on the Navy Dock at Sheerness, in the Thames Estuary, the plot constitutes the first known sea-level record of an entire (15-day) spring–neap cycle. By 1850, automatic tide gages were to be found in many large ports.

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The first to realize the practical potential of applying systematic harmonic analysis to tidal records was William Thomson, Lord Kelvin. In 1867, he appointed and chaired (on behalf of the British Association for the Advancement of Science) "a Committee for the purpose of promoting the extension, improvement, and harmonic analysis of tidal observations." Perhaps the most significant result was the eventual construc-tion of tide-predicting machines (TPMs) on the principle illustrated in Figure 2.

The effect of raising (lowering) any pulley in the upper (lower) row $h$ inches is to raise the stylus attached to weight $W$ by $2h$ inches. Thus, by causing each of the pulleys $i = 1, 3$, and 5 to stand $A_i \cos(\omega_i t \; - \; \theta_i)$ inches above its equilibrium position at times $t \; > \; 0$, and pulleys $i \; = \; 2, 4$, and 6 similar distances below, any rotation of the drum will cause the stylus to plot the sum of six harmonic functions of time. Moreover, since the desired vertical oscillations of the pulleys are easily produced by a combination of gears and bicycle chains, it is possible to propel such a device by turning a single crank. The giant Hamburg TPM was but an expanded (and electrically powered) version of Kelvin's machine.

Since the angular velocities $\omega_i$ are the same everywhere, only the amplitudes $A_i$ and phase lags $\theta_i$ peculiar to a given port need be measured in order to begin generating tidal predictions for that port. Typically, a few months worth of data is sufficient for rough estimates of these quantities, although fine-tuning can continue for several years.

Another activity inaugurated during the 19th century was the mapping of "co-tidal lines," the loci of simultaneous high water in the open oceans. Until quite recently, when the advent of satellite technology made direct measurement possible, it was necessary to infer the locations of such lines from measurements taken on shore. The first to attempt this was the Reverend William Whewell (1794–1866).

Cartwright describes Whewell as a "scientific philosopher" who occupied in succession the chairs of mineralogy and moral

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[*] In or about 1830, by an engineer named Henry Palmer.

philosophy at Cambridge University, before becoming master of Trinity College. In 1836, this jack-of-all-trades produced the last of several versions of such a map covering the greater part of the world's oceans. Adequate tidal records were not yet available for the Asian portion of the Pacific rim, or for extreme north and south latitudes.

In constructing his map, Whewell found it necessary to resolve seemingly inconsistent facts, including the previously mentioned southward advance of the hour of high water along the east coast of England, which presumably collides with its northward advance along the north coast of Europe. This he accomplished by supposing that certain families of co-tidal lines meet at isolated points in the North Sea. High water must occur at all hours near such a point, making it a point of no-tide. Other experts, including Astronomer Royal George B. Airy, refused to concede the existence of such points, and proposed alternative resolutions.

In time, the existence of "nodes" at which co-tidal lines meet in basins of no discernible tide was confirmed by direct measure-



**Figure 1**. *Plot of the first known (1831) sea-level record of an entire spring–neap tidal cycle. Both figures in this review are from* Tides: A Scientific History.

ment—for the first time in 1840 by a Captain Hewett of the British Royal Navy, who had careful soundings taken at 30-minute intervals throughout the hours of daylight (on two separate days, in particularly calm weather) from a longboat tethered at each end above a particularly shallow bank in the North Sea. Modern tidal maps indicate the locations worldwide of at least a dozen such nodes.

The best modern maps employ either colors or a gray-scale—overlain by white co-tidal lines—to reflect tidal amplitudes. The lines radiate outward from nodes located in the middle of (pale) basins of vanishing tidal amplitude, toward (dark) regions of extreme amplitude. The latter, of which Nova Scotia's Bay of Fundy is surely the most famous, are typically located near land. Unshaded maps employ two separate charts, one for co-tidal lines and the other for lines of equal amplitude. In Chapter 14, Cartwright displays a unified gray-scale map of all major tidal constituents, while the book's busy dust jacket affords glimpses of a colorized version.

Researchers devoted the first half of the 20th century to the furtherance of initiatives begun during the 19th. Notable among them were studies of the Arctic and of geophysical tides, including tidal motion in the atmosphere and in the Earth's molten core. Another recurring theme was the gradual slowing of the Earth's rotation about its axis—of which Laplace was already aware—often blamed on tidal friction. Genuinely new departures became possible only in the 1950s, with the availability of large-scale computers and (later) satellite technology. In particular, the near-complete mapping of the ocean floor, and the development of finite difference methods for solving PDEs, made it possible to solve the Laplace tidal equations with realistic boundary conditions and depth functions $D(\theta, \phi)$.

At first this could be done only for semi-enclosed segments of the ocean, such as the North Sea (and later the Atlantic), with the roughest of



**Figure 2**. *Mechanical tide-predicting machine.*

estimates used for the missing data along the interfaces with the global ocean. Later, as methods were refined and larger computers became available, the computational mesh was extended to the entire surface of the globe, eliminating the need for such estimates.

Three satellites—GEOSTAT, ERS-1, and the TOPEX/POSEIDON craft (launched jointly by France and the U.S. in 1992)—have ushered in a new era of global ocean monitoring by remote sensors. All bore altimeters of unprecedented accuracy (to 2–3 cm), supported by additional onboard instrumentation, large numbers of tracking stations, and elaborate computations of radial position. Each produced a continuous stream of high-quality data for more than three years. The results include complete global maps of all eight major standing-wave constituents of the tides—accurate to within a few centimeters. Cartwright himself was directly
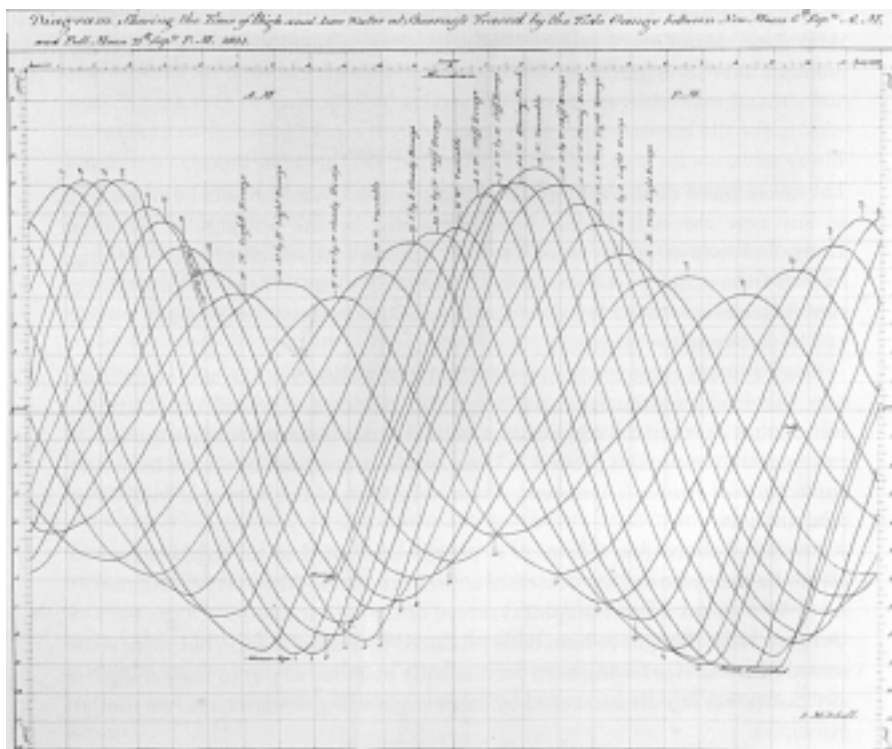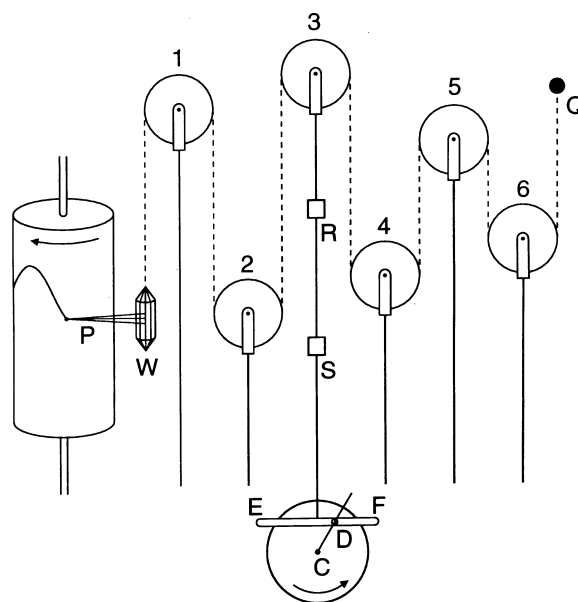
involved in the analysis of data from the first two missions and helped lead the international planning team for TOPEX/POSEIDON.

The book is by no means an easy read. Cartwright is uncompromising in his use of concise terminology, and often encyclopedic in his description of scientific activity. Yet for those whose curiosity is aroused by the title, the rewards are considerable. The author chronicles the entire history of a practically important branch of physical science, from the vaguest notions of the ancients to the nearly complete solutions now on hand. The resulting volume is well worthy of the luxurious "coffee-table" production given it by the Cambridge University Press. One hopes that a lower-cost paperback edition will be forthcoming.

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