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Community Lecture '99: Rapid-fire Tour Of BMX and Drag Racing

By James Case

Richard Tapia of Rice University gave the fourth annual I.E. Block Community Lecture in Atlanta, at the 1999 SIAM Annual Meeting, dedicating his talk "to Ed Block for the challenging early years and the major role that he played in making SIAM the wonderful organization that it is today." Drawing on an extensive collection of Tapia family photographs and memorabilia, Tapia surveyed a series of issues that remain controversial within the BMX (bicycle motocross) and drag-racing communities to illustrate the manner in which mathematical thinking and insight might contribute to their eventual resolution.

Fast Lane

How can a parent, coach, or technical adviser assist a quick and coordinated—but not particularly strong or fast—bicycle racer, such as Richard Tapia Jr. (who eventually did become a national age-group champion), to win races? For Tapia, quick starts were an obvious answer, and the audience was rewarded with footage of a youthful Richard Jr. exiting the starting gate constructed by Richard Sr.—who confesses to being a world-class mechanic—in the family's Houston backyard. But strategy as well as tactics would be needed to conquer the "running N"-shaped track (see Figure 1 on page 2) on which such races are run.

The first (steeply banked) turn in such a race is typically decisive, Tapia explained: Racers who emerge from it with a lead frequently win. Moreover, racers cursed with extreme outside starting positions are almost always forced to swing so wide on the first turn that—if they manage to remain on the track at all—they are all but eliminated from contention. It seems only fair, therefore, that each contest should consist of three separate "heats," with racers assigned to different starting positions in each.

The starting positions are numbered from 1 to 8, from the inside of the first turn out, as indicated (schematically) in the figure. In the past, starting positions were assigned by means of three separate random drawings. Tapia, however, was able to persuade racing officials to adopt a fairer procedure, one whereby no individual racer can ever be assigned to the outermost lane (#8) in successive heats.

He produced his design, a 3×8 table of starting positions, by placing numbers more or less randomly in the several cells, asking experienced racers which column of starting positions they would prefer, and modifying the table to eliminate clear favorites. When respondents were no longer able to identify a clear order of preference, iteration ceased and lobbying for acceptance began. The following table is now in widespread use:

With enthusiasm and energy to spare, Richard Tapia gave the audience at the I.E. Block Community Lecture in Atlanta a glimpse of the ways in which mathematical insight can pay off in the (exotic to many) worlds of BMX and drag racing. A few evenings earlier, also in Atlanta, Tapia had been co-guest of honor, with Rice University (and optimization) colleague John Dennis, at a joint 60th-birthday celebration.

	Racer							
Heat No.	1	2	3	4	5	6	7	8
1	6	7	4	5	2	3	1	8
2	2	4	7	1	6	8	5	3
3	5	2	3	7	6	4	8	1

The introduction of similar tables for races involving fewer riders has resulted in fairer contests nationwide.

Rounding the first turn, Tapia explained, the racer in the inside position faces a dilemma. If he clings to the inside, he takes no advantage of the banking of the turn and stands to be passed by someone who goes up higher, then "sling-shots" down off the bank and into the lead. Yet if he takes too high a line, he is vulnerable to being passed on the inside (A and B in Figure 1). Ordinarily, the rider in lane 1 can prevent being passed in this way by the rider in lane 2, on his immediate left. But so much concentration is required to do so that the lane 3 rider can often pass inside those in both lanes 1 and 2, while they are busy frustrating one another. That's why riders rejoice in being assigned to lane 3, while dreading assignment to lane 1. BMX racers, Tapia said, are well advised to "get out in front and protect your inside."

Gear Ratios

Tapia's second topic was automobile gear ratios, which he explained in terms of Figure 2. The key elements are those labeled "Ring Gear and Pinion." The pinion must turn about 3.5 times to make the ring gear turn just once. The precise number of turns required is *R/P*, where *R* and *P* denote the respective numbers of "teeth" or "cogs" on the ring and pinion gears. A low gear ratio corresponds—perhaps confusingly—to a high numerical value of the fraction *R/P*, while a high gear ratio corresponds to a low numerical value of that improper fraction. Although automobile buyers seldom inquire about gear (or axle) ratios, truck buyers frequently do. Low gear ratios ($R/P > 3.5$) compensate for poor gas mileage and high engine RPM on the highway, which tends to make the vehicle noisy—with good acceleration and good towing power. In contrast, high gear ratios ($R/P < 3.5$) compensate for poor acceleration and towing power with good gas mileage and quiet rides. Tapia offered a list of seven typical gear ratios, ranging from 3.23 = 42/13 to 4.56 = 41/9, pointing out that some are deemed more satisfactory than others.

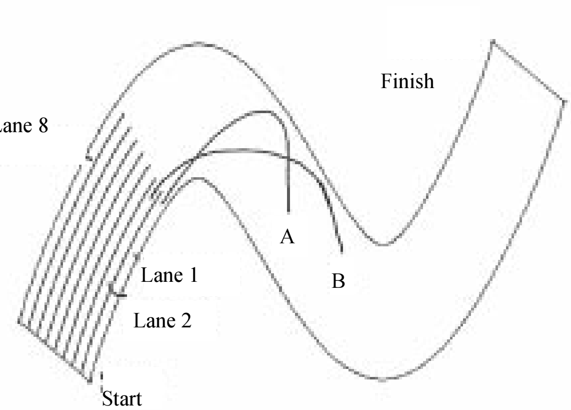

Figure 1. *On a "running N"-shaped track, the fairest possible lane-assignment scheme makes a big difference to bicycle racers.*

Tapia's list is by no means complete, since it excludes the values 3.1 = 31/10 and 3.44 = 31/9, popular on road racers, as well as 3.77 = 49/13 and 3.98, purportedly available on the 1999 Ford Taurus. The latter figure is suspect because a ratio in the interval [3.975, 4) would seem to require a pinion gear with at least forty cogs, significantly higher* than the number found in any of the standard configurations.

Although Tapia did not say so, the fact that 3.77 = 49/13 need not be discovered by trial and error. The following identities from "interval arithmetic,"

$$[753/200, 151/40) = 3 + [153/200, 31/40) = 3 + 1/(40/31, 200/153]$$

$$(40/31, 200/153 = 1 + (9/31, 47/153] = 1 + 1/[153/47, 31/9)$$

$$[153/47, 31/9) = 3 + [12/47, 4/9) = 3 + 1/(9/4, 47/12],$$

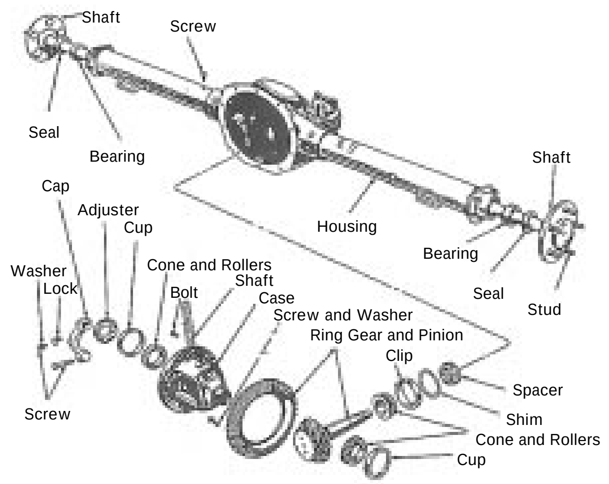

Figure 2. *Listeners were reminded of the importance of ring and pinion gear ratios in automotive design.*

together with the facts that $3.77 \in [3.765, 3.775) = [753/200, 151/40)$, and $3 \in (9/4, 47/12] \approx (2.25, 3.916]$, justify the continued fraction expansion $3 + 1/(1 + 1/(3 + 1/3)) = 49/13 = 3.7692\ldots \approx 3.77$. The "algorithm" terminates naturally the first time the rightmost interval in any line contains one or more integers. Interval arithmetic [1] is now widely used for scientific calculation, but remains underutilized in the classroom. The reader is invited to carry out the corresponding steps for $3.98 \in [3.975, 3.985) = [159/40, 797/200)$.

Tapia's first task was to explain why the standard ratios are relatively complicated, while simple ratios like 3.5 and 4.0 are never employed. This he did in terms of wear patterns. Consider a "designated tooth" P^* on the pinion gear, and let the teeth on the ring gear be denoted $R_1, R_2, \ldots, R_R$. If P^* is initially in contact with R_1, then those two teeth will again be in contact after $[R,P]$ additional pairs of teeth "kiss," where (R,P) and $[R,P]$ denote the greatest common divisor and the least common multiple of *R* and *P*, respectively. If $[R,P] = RP$, all potential osculations will have taken place during this time and will do so again repeatedly in the future. But if *R* and *P* have a common divisor $D \geq 2$, then at least half of all potential osculations will never take place.

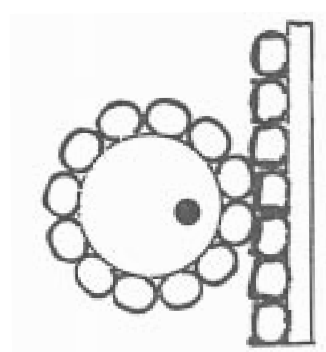

Figure 3. *An off-center pinion gear will result in "wobble" of the ring gear. Despite its formulation as an optimization problem, wobble does not seem to be a straightforward issue.*

Common divisors are probably unimportant if the gears are properly mounted in the first place. Yet they soon become important when (for instance) the pinion gear's line of axial symmetry fails to coin-cide with its axis of rotation, as in Figure 3.

In that event, some pinion teeth grind harder than others against the ring teeth they encounter, causing the ring gear to wear unevenly. The obvious remedy is to choose values of *R* and *P* that have no common divisors $D \geq 2$. This led Tapia to formulate his "uniform wear condition," $(P,R) = 1$. The

*To date, efforts to obtain an official description of the ring and pinion combination whereby Mrs. Case's new Taurus is equipped with an "axle ratio" of 3.98 have proven fruitless.

results of his informal poll reveal that designers are invariably aware of this requirement, though repair-shop personnel are not.

Having explained the relative complexity of gear ratios, Tapia was em-boldened to ask why some of them (such as 4.11 = 37/9) are more highly regarded than others (such as 3.42 = 41/12). His search for an answer led him only to the information that "the number of teeth on the pinion gear is 12, an even number."

Having observed that the ring gear will "wobble" if the pinion gear is off center (as in Figure 3), he defined the *wobble* $W(P,R)$ of a ratio R:P to be the modular distance from the tooth P_1 on the pinion gear initially in contact with R_1 to the next tooth (call it P_2) on that gear to make contact with R_1; he observed, however, that $W(P,R) = 0$ if and only if P divides R, thereby violating the uniform wear condition. This led him to formulate the following problem:

$$\min W(P,R) \quad \text{subject to } (P,R) = 1$$

and to observe that any gear ratio R:P for which $R \equiv 1 \bmod(P)$ solves the problem. Moreover, because $37 \equiv 1 \bmod(9)$, the ratio 4.11 = 37/9 uniquely minimizes the wobble† among all the gear ratios on his original list of seven. He then concluded that small wobble furnishes a plausible explanation for the apparent desirability of 4.11 as a gear ratio, but that large wobble does not seem to explain the *un*desirability of 3.42, since 3.55 performs rather well despite comparable wobble.

Controversial Records

Drag races are run on standard quarter-mile concrete tracks, by cars that reach speeds well in excess of 300 mph. (Fifty years ago, they were less than half as fast.) A clock, accurate to 0.001 seconds, is located at the finish line, and the driver who covers the carefully measured distance in the shortest time wins.

Drivers and fans are also interested in the intermediate times recorded by five additional clocks, located 60, 330, 660, 1000, and 1254 feet from the starting line. These intermediate times, along with the facts that $S(0) = 0 = S'(0)$, constitute raw data that can be used to estimate the function $S(t)$ representing the various intermediate distances covered during the various time intervals $(0,t]$. Together with the facts that $S(0) = 0 = S'(0)$, there are eight pieces of information to work with. The most frequently asked questions concern $S''(0)$ and $S'(T)$, where T is the unique instant for which $S(t) = 1320$ feet. $S'(T)$ is ordinarily estimated by calculating the average speed in the "time trap" located between the last two clocks. This provides a moderately accurate estimate, since drag racers are designed for rapid initial acceleration and have all but ceased to accelerate by the end of any ordinary race.

Suspicion persists, nevertheless, among aficionados that this "backward difference" procedure underestimates top speeds by as much as 2–3 mph. Controversy also surrounds the measurement of acceleration during the early stages of a race. Tapia obtained his estimates by interpolating the data points accumulated during a typical dragster run with (a) a cubic spline and (b) a polynomial of minimum order. The positively sloped curves in Figure 4 represent his (visually indistinguishable) estimates of $S(t)$ and $S'(t)$, while the generally negatively sloped curves are the (quite different) estimates of $S''(t)$ obtained by the two techniques. The well-known difficulty of estimating higher-order derivatives from low-order ones clearly manifests itself here.

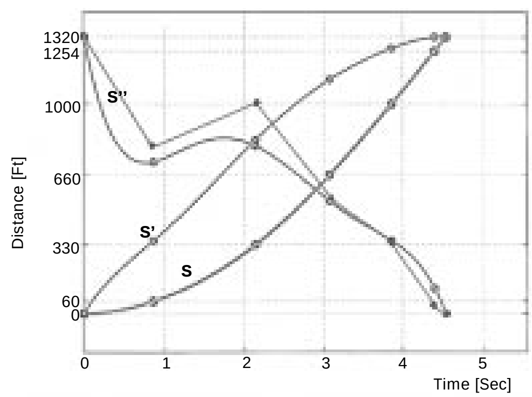

Figure 4. *To estimate acceleration during the early stages of a drag race, Tapia used a cubic spline and a polynomial of minimal order to interpolate the data points from a typical run.*

Accurate global estimates of $S''(t)$ will obviously have to await more detailed information. By utilizing polynomial fits of various degrees, however, Tapia obtained estimates of $S''(0)$ that vary from 5.07 g—an indisputable lower bound—to 7.10 g, and concluded that the true value probably lies between 6.5 and 7.0 g. Similar considerations led him to the conclusion that $S'(T)$ exceeds the average speed over the last 66 feet by no more than half, and perhaps as little as a sixth, of an mph. Both conclusions contradict drag-racing lore, which holds that reported values of $S'(T)$ are low by as much as 2–3 mph, while maximum accelerations range between 3 and 5 g.

Tapia went on to consider various sources of error in the measurement of top speed, concluding that clock resolution is by far the most serious. Indeed, for accuracy to 1 mph at 300 mph, a clock resolution of 0.0005 would be required. He then pointed out that Gary Ormsby, who was credited with 296.05 mph on September 9, 1990, was certainly going at least 292.21 mph and may actually have been the first to achieve 300. On the other hand Scott Kallita, who was credited with 308.64 mph on October 3, 1992, was certainly going 304.46 and may have been going as fast as 312.64 mph. In the interim, Kenny Bernstein, on March 12, 1992, Mike Dunn the same day, and Bernstein again a week later, were credited with speeds of 296.93, 297.12, and 301.70 mph, respectively. Hence, Kallita certainly drove faster than 300 mph, while any or all of the others may have!

Also described were long-running debates about the ultimate performance of drag racers, and a seemingly valid formula (attributed to an engineer named Vosburgh) relating acceleration to the coefficient of friction between rubber and road, from which editor Barney Navarro of *Rods and Customs* magazine deduced in 1953 that wheel-driven hot rods can go no faster than 167 mph.

†He neglected to mention, however, that any gear ratio R:P for $R \equiv -1 \bmod(P)$ also solves the given problem, and to explain why 4.11 is preferred to 3.91 = 43/11, a ratio that might well have been added to the original list.

Loading Questions

Tapia's final topic for the evening concerned the proper "tongue weight" for a trailer towed behind a car or pick-up. The tongue weight of an ordinary single-axle trailer, measuring d_0 feet from axle to hitch, loaded with weights $w_1, \ldots, w_n$ placed at (signed) distances $d_1, \ldots, d_n$ before the axle—distances aft the axle being treated as negative—is, by definition, $w = \sum_i w_i(d_i/d_0)$. It can ordinarily be determined with the aid of a bathroom scale.

The dangers of excessive or insufficient tongue weight were explained, and the speaker's experiences loading his own "rig" were recounted with the aid of additional family photographs. This part of the lecture resonated with all the former rum-runners in the audience, or with anyone who had ever been forced to contend with unbalanced or shifting loads, and revealed some surprisingly counterintuitive aspects of a seemingly simple subject. It was a fitting end to a lively, informative, occasionally heart-warming, and thoroughly entertaining community lecture.

References

[1] James Case, *Interval arithmetic and analysis,* Col. Math. J., 30–2 (1999).

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