from *SIAM News*, Volume 31, Number 2

Making Waves: Solitons and Their Optical Applications

Bonnie and Bill Kath

First Place SIAM Writing Contest

Soliton technology has recently achieved close to pop science status as its proponents seek to put it to use for transporting vast amounts of information—the stuff that gives our era its name—farther and faster. It could well become one of the fundamental technologies in the current communications revolution.

It was during the Industrial Revolution, however, that this phenomenon was first noticed and studied. A Scottish engineer by the name of John Scott Russell had set out to create a more efficient hull design for canal boats (a 19th-century forerunner, perhaps, of current efforts to speed packets of information along fiber-optic cables). One day in August of 1834, he stood beside Union Canal near Edinburgh to observe the movement of a boat being pulled by two horses. As the rope pulling the boat snapped and the boat's movement halted, its prow dropped back down and Scott Russell saw a large mass of water, a smooth, solitary wave, gather around the prow and continue rapidly down the channel.

Surprised and intrigued, he followed on horseback and noticed that the wave held its shape and only very gradually diminished in height. He lost sight of it after a mile or two but was so taken with this observation that he built a 30-foot wave tank in his back yard to study the phenomenon further. Ten years later, he reported his observations to the British Association for the Advancement of Science, calling what he had observed the "wave of translation." Scott Russell considered that day back in 1834 "the happiest day of my life" [4], but his discovery was ignored by all but one or two people, who felt compelled to prove him wrong in the scientific literature. After all, it was common knowledge that waves could not behave in this way.

A Flurry of Mathematical Activity

Vindication came from two independent camps, both of which were attempting to explain the movement of shallow water waves. Boussinesq's equation in 1872 and the Korteweg–de Vries (KdV) equation in 1895 proved that solitary waves were, indeed, theoretically possible. Both equations describe the evolution of the wave height $\eta(x, t)$, but the KdV equation,

$$\frac{\partial \eta}{\partial t} + 6\eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{1}$$

is the simpler of the two [1, 3] and has the solitary wave solution

$$\eta = 2a^2 \operatorname{sech}^2 a\left(x - 4a^2 t\right), \tag{2}$$

where $2a^2$ is the wave amplitude and $4a^2$ is its velocity.

It would be another 70 years before these findings were taken any further. In 1965, Martin Kruskal and Norman Zabusky studied the KdV equation numerically and determined that nonlinear solitary waves could occur naturally, given the right conditions. They also obtained a surprising result: Even though the equation is nonlinear and two solitary waves propagating at different speeds should thus interact

Judges Award Two Prizes In First SIAM Writing Contest

"It is often said that interest in science is stimulated by its many practical applications. Nothing could be further from the truth."

Writing in a recent issue of *The New York Times Book Review*, John Durant, a professor of public understanding of science at Imperial College, London, was making the point that books on "the big questions"—cosmology, evolutionary biology, and consciousness—sell far better than books on chemistry, ecology, and thermodynamics. "It is not that we doubt the importance of catalysis, the carbon cycle and combustion," he wrote; "it is just that we find these things less gripping than the Big Bang, the death of the dinosaurs and the search for a coherent theory of mind."

All undoubtedly true, but where does this leave the scientists and mathematicians whose work is concerned with practical applications? Mainly, it seems, with a task that is very difficult. Just ask the nine authors who entered SIAM's first writing contest.

Each author took the time to prepare and submit a well-thought-out presentation of an interesting application of mathematics. They were writing for their peers—the SIAM membership and other readers of *SIAM News*—and their challenge was to present an interesting, novel approach to an important application of mathematics.

The contest judges had mixed feelings; almost all found the applications themselves interesting and important. Most of the judges commented that the authors (not surprisingly) were clearly expert at writing research papers for peer-review journals but were venturing into new territory as newspaper writers. (Authors who could write for an audience more broad than the SIAM membership was a long-term goal of the SIAM board in initiating the contest.)

In the end, the judges selected the paper "Solitons and Their Optical Applications," by Bonnie and Bill Kath, as the first-prize winner. The article, complete with colorful history, rich and ongoing mathematical developments, and an element of suspense, begins on page 1 of this issue. The authors have received \$1000. The judges also chose to award an honorable mention, to Mark Frantz, whose submission is titled "How to Look at Art"; Frantz, whose article will appear in an upcoming issue of *SIAM News*, receives \$500.

SIAM News is pleased to publish both of these articles and hopes that they will inspire many others to write about their work for a more general audience, possibly for the next SIAM writing contest.

strongly with one another, the interaction is only temporary and the waves quickly recover their original shapes and velocities. These elastic collisions were similar enough to those of colliding elementary particles that Kruskal and Zabusky coined the term "soliton" for such waves.

Within the applied mathematics community, work then advanced rapidly. In 1967, the team of Gardner, Greene, Kruskal, and Miura devised the method for the exact solution of the KdV equation now known as the inverse scattering transform. In work considered by many to be a mathematical tour-de-force, they showed that if the solution of the KdV equation is treated as the potential function in the scattering problem

$$\frac{\partial^2\psi}{\partial x^2}+\left[\lambda+\eta(x,t)\right]\psi=0, \qquad (3)$$

then even though η changes with time, the spectrum (defined by λ) is time independent. In addition, the associated spectral data evolve in time in a linear and very simple manner. The final step is to recover the potential η (and thus the solution of the KdV equation) from the scattering data, a result that had been worked out in the early 1950s by Gelfand and Levitan. The resulting inverse scattering transform made it possible to find a number of exact analytic solutions in closed form and demonstrated explicitly the elastic collisions observed by Zabusky and Kruskal (see Figure 1).

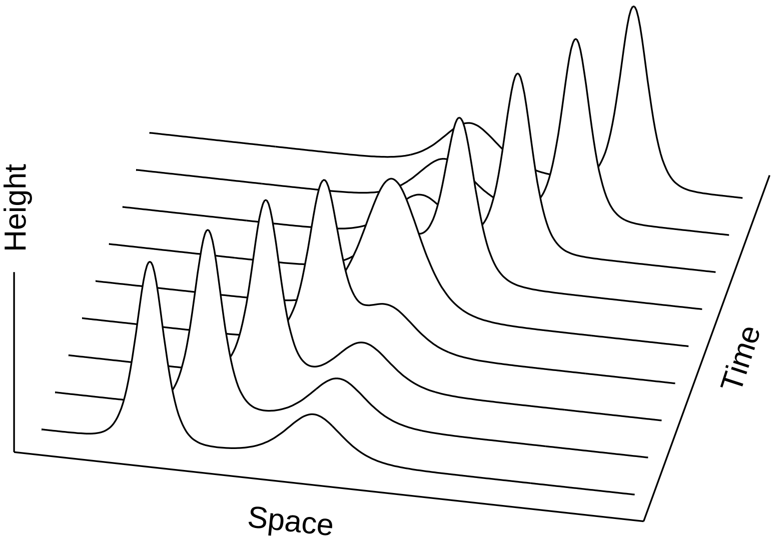

Figure 1. *Soliton collision: A larger and faster pulse overtakes a smaller and slower one. The two solitons emerge from the collision with their identities intact. The wave height profile η is shown at several times.*

This amazing discovery was shown not to be a fluke when Zakharov and Shabat, using an elegant formulation developed by Peter Lax, were able to solve the nonlinear Schrödinger equation, which describes the evolution of a general wave packet's slowly varying envelope. Still more complicated nonlinear equations were subsequently dispatched when Ablowitz, Kaup, Newell, and Segur showed how to make the method more systematic with a procedure now known as the AKNS method.

This flurry of mathematical activity caused the scientific community at large to sit up and take notice. (Or, perhaps, the term "soliton" captured researchers' imaginations better than Scott Russell's "wave of translation" had.) During the past 30 years, soliton research has been conducted in fields as diverse as particle physics, molecular biology, quantum mechanics, geology, meteorology, oceanography, astrophysics, and cosmology.* But the area of soliton research at the forefront right now is the study of optical solitons, where the highly sought-after goal is to use these nonlinear pulses as the information-carrying "bits" in optical fibers.

Compensating for Dispersion

Any information-carrying signal, by necessity, contains components from a range of frequencies. Because the propagation speed of a wave is frequency-dependent, the transmitted pulses tend to spread (an effect called dispersive spreading) and the signal, in turn, tends to break up. Since the inception of optical fibers, researchers have been looking for ways to combat dispersive spreading because of the severe limits it imposes on the capacity of optical communication systems.

An exploration of the details of dispersive spreading can help us understand this limitation. The dispersion produced by an optical fiber can be measured in units of picoseconds squared per kilometer (ps^2/km). "Standard" single-mode optical fiber (Corning SMF-28**), for example, has a dispersion parameter of about 20 ps^2/km at a wavelength of 1.55 microns (one of the two main communication wavelengths). The amount of dispersive spreading can be estimated as the square root of the product of the dispersion parameter and the propagation distance. For standard fiber, the dispersive spreading after 500 km is then roughly $(200\times 500)^{1/2}=100$ ps. If the transmitted pulses are approximately 100 ps wide (corresponding to a "bit-rate" of 1 Gbit/s if the pulses are spaced 10 widths, or 1 ns, apart), they will have spread by an amount equal to their own widths after only 500 km. Since this effect increases with the propagation distance, after 10,000 km (the distance across the Pacific from the U.S. to Japan) the pulses will have spread to 4.5 times their original widths. Pulses that have spread this extensively will overlap, making it increasingly difficult to separate them from one another; the result is degradation of the signal.

Seeking a way around this difficulty, Akira Hasegawa and Fred Tappert proposed in 1973 that a nonlinear dependence

*Some of these applications were explored at a 1995 conference celebrating 100 years of the KdV equation. A particular highlight was the re-creation of Scott Russell's observations, which can be viewed on the Web at http://www.ma.hw.ac.uk/solitons/press.html.

**See http://www.corningfiber.com/fiber.htm.

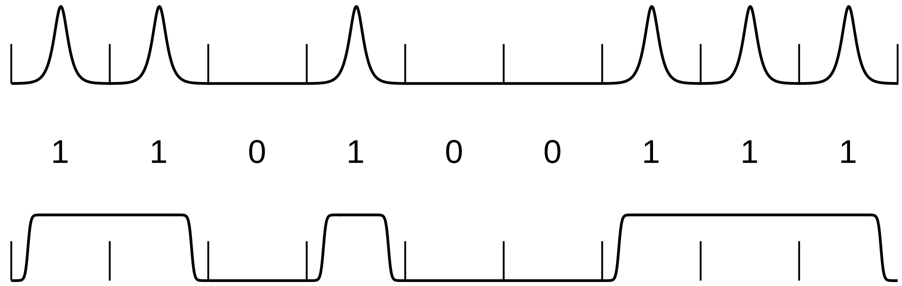

Figure 2. *Soliton (top) and NRZ (bottom) encoding of the binary data sequence 110100111. The vertical lines mark the extent of each bit's timing window.*

of the index of refraction on the light intensity could be used to compensate for dispersion. Essentially, light is guided by total internal reflection in a region where the index is higher than that of its surroundings; optical fibers use an index difference to keep propagating light trapped in the higher-index core region, for example. When a light pulse is intense enough to increase the index of refraction on its own (which became possible with the invention of the laser), the pulse becomes self-trapping and no longer disperses.

Hasegawa and Tappert showed that the nonlinear Schrödinger equation studied by Zakharov and Shabat (but with space and time interchanged, as appropriate for optical fibers) [2], namely

$$\frac{\partial u}{\partial z} = i\frac{1}{2}\frac{\partial^2 u}{\partial t^2} + i|u|^2 u, \tag{4}$$

is the appropriate equation for describing the nonlinear propagation of light in optical fibers. They performed a number of computer simulations demonstrating that nonlinear pulse transmission in optical fibers would be stable. These theoretical findings were intriguing, but such pulses were impossible to observe experimentally, because light transmitted through optical fibers was still experiencing too much loss.

By the early 1980s, however, fiber-optic technology had caught up somewhat, and Linn Mollenauer, Roger Stolen, and Jim Gordon were able to observe soliton propagation in the lab. As mathematical results continued to appear, this group at Bell Labs was experimenting intensively with optical solitons, looking for ways to use them in long-distance telecommunication systems. Ironically, while their work was enthusiastically received in scientific circles, it seemed that the practical application could not be quickly realized, and they were told by the head of research, Arno Penzias, to desist. Doggedly, they persisted until their results were so compelling that Penzias apologized and publicly praised the work [5]. Mollenauer's group has since achieved several long-distance and speed records for optical soliton transmission.

Erbium-doped Fiber Amplifiers

One of the key technological developments that made such soliton experiments possible was the invention of the erbium-doped fiber amplifier (EDFA) in 1987. Researchers at Southampton University, NTT, and Bell Laboratories discovered that when a short length (e.g., a few meters) of optical fiber is doped with the rare-earth element erbium and pumped with light energy from a diode laser, it becomes an optical amplifier. Up to this point, optical transmission systems had used electronic repeaters, which convert light to electrical energy, clean it up, amplify it, and finally, using a semiconductor laser, launch it on its way again. The cost of such electronics becomes prohibitively expensive at high bit rates, though, which imposes a practical limit on the information capacity of repeater-based systems. Because of the simplicity and large bandwidth of EDFAs, they quickly became the amplifier of choice in transmission systems.

The development of EDFAs also breathed new life into other, more traditional, transmission schemes and greatly accelerated the growth of the global optical fiber network, including the long-distance segments deployed under the world's oceans.*** Since the installation cost for undersea cables can exceed $1 billion and the systems technology must work correctly the first and every time, it is important to employ the fastest and most robust scheme available in designing such telecommunication links. A race therefore began between solitons and the other schemes for the long-distance speed record.

NRZ Versus Solitons

The main competition in this race has come from a data-encoding scheme known as non-return-to-zero, or NRZ, that was developed for more traditional, linear transmission systems. As shown in Figure 2, the main difference between encoding digital signals with solitons and with NRZ is that in the latter case if two 1s are close together the signal intensity doesn't drop back to 0 between the individual bits as it does with solitons. For NRZ, however, the nonlinearity inherent in optical fibers is a problem that can distort signals at high speeds.

Relatively quickly, a team led by Neal Bergano at AT&T learned how to increase the bit rate of NRZ-encoded signals

***See *International Submarine Cables of the World*, located on the Web at http://elaine.teleport.com/~ptc/iscw/iscw.shtml.

to 5 gigabits of data per second over a distance of 9000 kilometers. (Roughly speaking, 5 gigabits is about the amount of information stored on one CD-ROM disk or contained in the text and figures of a set of encyclopedias.) They dealt with the problem of dispersion by employing newer fiber (known as dispersion-shifted fiber) that is constructed to move the zero-dispersion wavelength (the wavelength at which the dispersion vanishes) close to 1.55 microns, where erbium-doped amplifiers operate. Their results prompted AT&T to retain NRZ as the data-encoding scheme for undersea transmission systems.

At this point, solitons were able to operate at 2.5 Gbit/s over 9000 km, but at 5 Gbit/s were able to survive a distance of only 6000 km. Jim Gordon of Bell Labs and Hermann Haus of MIT had already given an explanation for the difficulty, showing that it is a fundamental result of the ways in which EDFAs work with solitons.

When a doped optical fiber is pumped with light from a diode laser, the erbium atoms absorb energy and are put in an excited state. A photon of signal light entering the EDFA can then stimulate an excited erbium atom to coherently emit an additional photon, thus amplifying the signal. As Einstein had shown, however, stimulated emission must always be accompanied by a certain amount of spontaneous emission. Such randomly released photons act as noise and are added to the propagating signal. In the case of solitons, the added noise produces fluctuations in the pulses' positions (now known as Gordon–Haus timing jitter) and leads to transmission errors at high bit rates or large enough distances. At 5 Gbit/s, the theoretical distance limit turns out to be roughly 6000 km, precisely what was observed by Mollenauer.

Another piece of soliton mathematics then entered the picture. Working independently, Kodama and Hasegawa, and Antonio Mecozzi, John Moores, Hermann Haus, and Yinchieh Lai at MIT, predicted that the maximum propagation distance of optical solitons could be increased with the addition of optical filters to the transmission system. Such filters remove some of the noise added by the amplifiers, extending the distance over which solitons can propagate. As predicted, after adding filters to the experiment, Mollenauer was able to propagate a 5-Gbit/s signal for a distance well beyond the required 10,000 km. He also improved on the idea by varying the filters' central frequencies linearly with distance along the fiber, something he called "sliding-frequency guiding filters." With this modification, he was able to send a 10-Gbit/s signal more than 20,000 km.

Dispersion Management

Neal Bergano and his NRZ team were not idle during this time, of course. Although they were not able to increase the rate of an NRZ-encoded bit stream beyond 5 Gbit/s, they were able to increase the overall bit rate by employing a technique known as wavelength-division multiplexing (WDM). Much as in cable TV, they chose several wavelengths as different channels and simultaneously transmitted a separate 5-Gbit/s data stream in each one. In 1996, Bergano and his team were able to send 32 separate 5-Gbit/s channels, for a total of 160 Gbit/s, over 9000 km.

Systems employing solitons have also been able to exploit WDM to increase the total bit rate, but not yet to the extent obtained with NRZ. Linn Mollenauer's team has transmitted eight separate 10-Gbit/s channels using solitons, for a total of 80 Gbit/s, and Masataka Nakazawa of NTT Laboratories has achieved five separate 20-Gbit/s channels, for a total capacity of 100 Gbit/s.

The race is not over yet. Neal Bergano's NRZ experiments worked so well in part because of a technique known as dispersion management. Essentially, the idea is to concatenate two or more fibers with different dispersion parameters to form a system with periodically varying dispersion. In this way it is possible to have both high local and low average dispersion in the system. The high local dispersion tends to reduce an effect known as four-wave mixing (which tends to distort signals and produce intersymbol interference)—the high dispersion disrupts the phase-matching of the different optical frequencies making up a signal, thus reducing the interactions between them. Moreover, the low average dispersion reduces its net cumulative effects over long spans of optical fiber.

Although dispersion management was invented for use with NRZ signals, a number of researchers have been investigating its use in soliton systems, requiring that the mathematics of solitons be extended even further. The first results were obtained by Akira Hasegawa and Yuji Kodama in 1991, when they applied asymptotic averaging methods using Lie transforms to the nonlinear Schrödinger equation with rapidly varying coefficients,

$$\frac{\partial u}{\partial z} = i\,\frac{1}{2}\,\sigma\!\left(\frac{z}{\varepsilon}\right)\frac{\partial^2 u}{\partial t^2} + ig\!\left(\frac{z}{\varepsilon}\right)|u|^2 u = 0. \tag{5}$$

Here $\sigma(z/\varepsilon)$ is the periodic dispersion, $g(z/\varepsilon)$ results from oscillations due to loss and periodic amplification, and ε is a small parameter used to indicate that these variations are rapid.

Kodama and Hasegawa's results showed that an averaged soliton, which they call the "guiding-center soliton," propagates in such a system. Since the effective dispersion parameter in this case is just the average over one period, it can be reduced by proper dispersion management. This decreases the total amount of Gordon–Haus timing jitter. More recently, a group at Aston University in Manchester, England, composed of Nick Smith, Finlay Knox, Nick Doran, Keith Blow, and Ian Bennion, showed that strong dispersion management enhances the power of optical solitons, further reducing the effects of the Gordon–Haus timing jitter.

The mathematical theory of nonlinear wave equations with rapidly varying coefficients like equation (5) is still not fully resolved, and is a subject of ongoing research around the world. A primary goal is to understand the pulse behavior when the variations are strong; this would provide a foundation for explaining a number of dispersion-management experiments that have been performed. But an even bigger objective, and a wide-open problem, would be to formulate a theory that would be capable of predicting the behavior of WDM systems with many channels.

Even if optical solitons don't eventually win this long-distance race, a number of spin-offs are now firmly entrenched technologically or are being heavily investigated. For example, lasers constructed almost entirely of optical fiber (with erbium-doped fiber as the gain medium) produce an endless stream of solitons as their output. A number of such "soliton lasers" can be purchased commercially. In addition, since today's high-speed computer networks deal with information packets (i.e., digital data organized into sequential groups), they tend to work better with single-channel transmission systems than with WDM-based systems. Because solitons have been able to surpass the 5-Gbit/s limit, they are currently the best candidate for ultra-high-speed packet-switched optical networks.****

Whatever the outcome of these investigations, there is no doubt that the study of solitons, both optical and otherwise, has inspired tremendous amounts of research in many disciplines, and has been a shining example of the potential benefits to be gained by a close interaction between science and mathematics.

References

[1] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

[2] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications*, Oxford, 1995.

[3] A.C. Newell, *Solitons in Mathematics and Physics*, SIAM, Philadelphia, 1985.

[4] John Scott Russell, *The Modern System of Naval Architecture*, Vol. 1, Day and Son, London, 1865, 208.

[5] *The Wall Street Journal*, June 25, 1991.

Bill Kath is a professor in the Department of Engineering Sciences and Applied Mathematics at Northwestern University. Bonnie Kath is a freelance writer based in Evanston, Illinois.

****MIT's Lincoln Laboratories is one of the key centers where this work is being performed; see http://www.ll.mit.edu/aon/UCaonTDM.html for more information.