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Strength Through Connections at IMA

By Barry A. Cipra

Work with great speed. Have your energies alert, up and active.

—*Robert Henri,* The Art Spirit

Avner Friedman, the former director of the Institute for Mathematics and Its Applications at the University of Minnesota who is renowned for his energetic efforts at IMA and elsewhere, recently celebrated his 65th birthday. The occasion, shared with his wife, Lynn, brought friends and colleagues to IMA for a workshop in honor of the event. The title of the workshop was appropriately Friedmanesque: "Pure, Applied, and Industrial Mathematics: Strength Through Connections."

Although no longer running IMA, Friedman is still an active member of the Minnesota mathematics department; also, he is director of the Minnesota Center for Industrial Mathematics. Established in 1993 within the university's School of Mathematics, the center provides graduate students with broad training in problem solving and modern analytical and computational methods for use in industry and business.

Talks at the workshop ranged from "Spinning and Weaving" (on mathematical modeling in the textile industry, given by Hilary Ockendon of the Oxford Centre for Industrial and Applied Mathematics) to "What Can We Learn from International Comparisons in School Mathematics?" (Zalman Usiskin of the University of Chicago). The Friedmans were also honored at a banquet, where long-time friends shared reminiscences of the couple. (Walter Littman, who teaches a course in mathematical modeling at the University of Minnesota, recalled an evening he and Avner had spent with the famous Russian mathematician Besicovich, who taught them the correct way to drink vodka.) The Friedmans' oldest daughter, Alissa, led a trio in an adaptation of the Beatles tune "When I'm Sixty-Four," with the refrain "Will you still hold me, nag and cajole me, when I'm 65?"

Overcoming Obstacles

A curve does not exist in its full power until contrasted with a straight line.

—*Henri*

Luis Caffarelli of the University of Texas gave a talk titled "The Obstacle Problem Without Obstacle." Loosely speaking, the obstacle problem is concerned with the design of an efficient ramp for approaching a hill. More precisely, given a domain D and an "obstacle function" f, the problem is to find a function u that minimizes the integral of $\|\nabla u\|^2$ over D, subject to the usual condition that u vanishes on the boundary of D and the "obstacle condition" $u \geq f$. (The problem requires that $f \leq 0$ on the boundary of D.)

In a typical obstacle problem, the minimizer u_0 agrees with f on part of the domain and differs near the boundary—that is, the function $w = u_0 - f$ vanishes on a subset of D and is positive elsewhere. It's obvious, though, that the minimizer is oblivious to small changes in the obstacle in the region where they differ; in that region, u_0 is harmonic, i.e., $\nabla u = 0$. The upshot is that much of the mathematical theory of the obstacle problem can be formulated in terms of functions w that vanish on a subset of a domain D and have the property $\|\nabla w\| = 1$ on the complement.

From this point of view, however, the original requirement $w \geq 0$ has disappeared—and may even be violated. In short, a good theory doesn't let obstacles get in its way.

Emmanuele DiBenedetto of Northwestern University described a different kind of obstacle problem: characterizing two-dimensional cracks in three-dimensional bodies. DiBenedetto and Giovanni Alessandrini of the University of Trieste have shown that, under reasonable conditions, exactly two distinct measurements suffice to identify a crack uniquely. Their work extends earlier results of Friedman and Michael Vogelius of Rutgers University, who studied one-dimensional cracks in two-dimensional regions.

A crack is a discontinuity in the conductivity of an object—in, that is, the ease with which current flows through it. Cracks can be either conducting or insulating. (In the latter case, the electric field vanishes at the crack.) A measurement is obtained by fixing electrodes at two points, P and Q, on the surface of the body

Artful Leader

Robert Henri (pronounced "hen-rye") was a key figure in American art in the late 19th and early 20th centuries. He was a leader of the New York Realists, later dubbed the Ashcan School for their often gritty portraits of urban life. Henri courageously rejected the traditional niceties of "academic" art and encouraged others to do the same. Had he been a mathematician, Henri would no doubt have been drawn to industrial mathematics.

Although a superb artist in his own right, Henri was most influential as a teacher. Among his students were such famous artists as George Bellows, Edward Hopper, and Stuart Davis. *The Art Spirit*, from which the quotations in the accompanying article are taken, is a compilation of Henri's advice to students. It is still in print today.

and then recording the electric potential on some portion of the surface. Provided that the cracks stay away from the surface and do not intersect, DiBenedetto and Alessandrini's theorem asserts that just two measurements are required to identify them. In the conducting case, it suffices to use two positions, Q_1 and Q_2, for the second electrode; in the insulating case, the positions of both electrodes must be changed, in such a way that the four positions P_1, Q_1, P_2, Q_2 do not lie in a single plane.

The new results are not just generalizations of the two-dimensional theory, and they don't supersede the earlier proofs, DiBenedetto explains. The three-dimensional proof doesn't work in two dimensions, he says, but rather depends on an important topological property in three dimensions: The excision of a crack from a simply connected, three-dimensional domain leaves the domain simply connected. That's simply not true in two dimensions. Planar geometry, it seems, is not all it's cracked up to be.

Film School

The effort to put down what we actually know—that is, what we can carry away with us—is often a revelation of the very little understanding we had in the presence of the model.

—Henri

Peter Castro of Eastman Kodak Company described efforts to use mathematics to improve photographic film. Color film, he explained, is a complicated physicochemical system, with layers of silver-halide grains and color-forming oil droplets separated by filters for different colors of light. The activation of a grain by incoming light is, ultimately, a quantum-mechanical process: Photons create electron–hole pairs, which wander in Brownian fashion until the electrons are captured by traps at the surface and neutralized by mobile silver atoms.

Conceptually, the process can be modeled as a Markov chain, but the number of states, on the order of 10^{50}, is prohibitive for anything but Monte Carlo methods. And that's just to study a single grain. A full-fledged simulation of the exposure process ought to survey hundreds of grains, starting with a study of the propagation of light through a random medium in which not only the particles, but also the distances between them, are roughly the size of the wavelengths of the light they're scattering.

If exposure is a complex physical process, film developing is an almost unimaginably complicated chemical event. In outline, the developing agent acts on the photon-altered silver-halide grains, causing them to grow silver "whiskers." These reaction by-products, in turn, react with the color-forming oil drops; the result is a tiny bloom of color near where, at one time, a photon of that color's wavelength chanced to pass. But whereas the photon was following Maxwell's equations, the chemical creation of color depends on a set of reaction–diffusion equations, again complicated by the random medium.

The developing process must carefully limit the extent of the reaction–diffusion process. In aesthetic terms, the process can be viewed as a competition between the desire for sharpness (you don't want all your photos looking like Renoir masterpieces) and the desire to avoid graininess (you don't want them looking like Seurats, either). The technological answer is to include chemical inhibitors that allow just the right amount (more or less) of oil droplet activation.

Traditionally, improvements in manufacturing processes in the photographic film and other industries have proceeded in "Edisonian" fashion, Castro says, with trial-and-error experimentation playing the key role. But "learning by experience is no longer a viable way to do business," he says. There are compelling economic reasons for using mathematical models to guide innovation. In one case, he observes, an optimization code for propagation phenomena was able to reduce the number of production-scale runs for a new process from 30 to 3. The savings for the immediate project was only a few million dollars—small potatoes for a company the size of Kodak—but the code can propagate similar savings through other processes. As Everett Dirksen once said of the federal budget, "a billion here and a billion there, and pretty soon you're talking about real money!"

There are big bucks in textiles, too. After all, there are nearly six billion people on the planet, most of whom wear clothes. Hilary Ockendon described some of the ways in which mathematical models can help unravel the complications of fiber processing.

In particular, there is a step in the creation of synthetic fibers in which thousands of strands of fiber are pulled through a liquid bath. As it moves through the bath, the "tow" of fibers is stretched: Rollers at the two ends of the bath turn at different rates, with the exit-end rollers turning faster. This causes the fibers' velocity to vary throughout the bath. What engineers need to be able to predict is the fluid flow in and out of the tow and, especially, the stability of the flow.

The step in fiber processing known as crimping also poses mathematical challenges, Ockendon explains. Roughly speaking, crimping is what will eventually give the fabric its thickness and texture. Fibers emerge from between two rollers and enter what's called a stuffer box—essentially, two flat plates, with a weight pressing them together. The fibers enter at "high" velocity (about a meter per second) and exit more slowly. As a result, they get balled up—crimped—in between.

"Primary" crimping begins just beyond the rollers, where the fibers buckle into the sinusoidal-type patterns seen in elastic beams—except that the crimp becomes permanent when plastic deformation sets in at a critical value for the strain. Secondary crimping—a larger-scale buckling of the primary crimp—then takes over. Finally, frictional forces in the stuffer box, best modeled by a delay integral equation, cause the fibers to pile up even further. A detailed mathematical description, Ockendon says, can help textile manufacturers understand—and presumably improve—these processes.

Mathematical models, she adds, also come into play at other stages in the textile and apparel industries, such as quality assessment (spotting flaws in, say, expensive cashmere), the generation of cutting layouts (a classic optimization problem), and color matching. The textile industry could easily keep scores of mathematicians busy in high-tech, computational sweatshops.

Upscale Mathematics

Don't demonstrate measure but demonstrate the results you may get from the employment of measure. There is geometry in all good expression.

—Henri

In a talk on stochastic fluid dynamics, James Glimm of the State University of New York at Stony Brook described recent advances in "upscaling" and other techniques for solving previously intractable flow problems.

Stochastic methods in general have gone from taboo to totemic in engineering circles. The change in attitude, Glimm points out, stems partly from the hostile takeover of finance by stochastic models. Oil companies, for example, are no longer satisfied with "definitive" answers to questions regarding oil recovery: They increasingly view such results as financial options, and want to know the risks that attend the rewards. (Wendell Fleming of Brown University also spoke at the workshop on stochastic models in finance.) The stochastic approach now permeates engineering, especially when it comes to problems involving mixing.

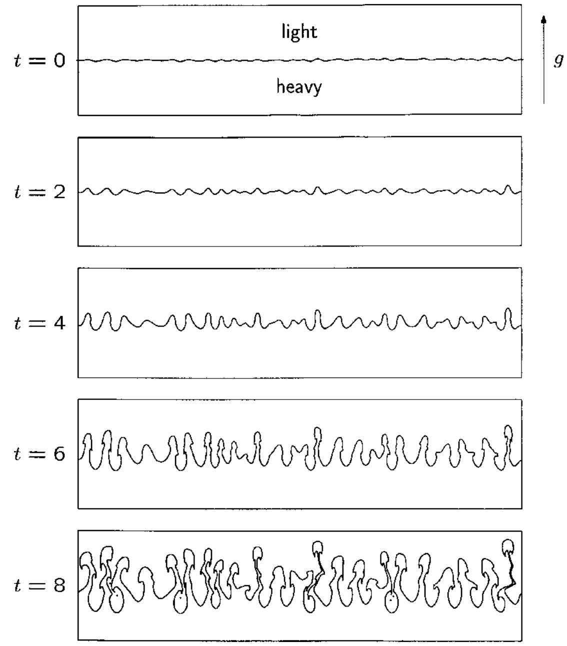

Figure 1. *Snapshots of the interface between polytropic gases with a density ratio of 2:1 in a front-tracking simulation with mesh dimensions of 800 × 800. (From Yu-Pin Chen, 1995.)*

Upscaling can be thought of as the multigrid method run backward, Glimm explains. In the multigrid method, problems are solved initially at a coarse level. These rough answers are then polished at finer and finer levels, determined in part by the solution itself. In upscaling, data at the finest scale are used to prepare equations that apply to the coarsely gridded solution. In short, it's a way to identify the relevant parameters when averaging over the actual, microscale behavior of a system to obtain a useful, macroscale description. (Of course, Glimm points out, "one person's micro is another person's macro.")

Glimm and colleagues at Stony Brook and Los Alamos National Laboratory have used upscaling in a study of a fluid mixing layer based on the two-dimensional Euler equations. At the small scale, they formulate the "exact" microphysics and solve the equations numerically, using a front-tracking method (see Figure 1). These direct numerical simulations, run on a massively parallel supercomputer system at Stony Brook called Galaxy, generate a stochastic ensemble that provides input for the upscaling step. The results they have obtained, Glimm says, agree extremely well with experimental results in mixing. He predicts that the fluid mixing problem will soon be solved and sees a bright future for other problems in multiphase flow.

"We think we have something that's sufficiently correct that it can actually be used," Glimm concludes.

Statements like that are music to Avner Friedman's ears.

Errant Errands

If the end is a revelation of a profound mystery of nature the sequences must be equal to it, for the revelation is written in every part of the construction.

—Henri

There's more to life than PDEs, of course. Shmuel Winograd of IBM's T.J. Watson Research Center weighed in with a description of recent work done with IBM colleague Barry Trager on algorithms for a new generation of Reed–Solomon codes. (Winograd also recalled one of the "dangers" of knowing Friedman: Any discussion of a potential topic for an IMA workshop led inevitably to the question, Will you help organize it?)

The general theory of Reed–Solomon codes is based on the algebra of polynomials over finite fields. In practice, such codes currently use the finite field $GF(2^8)$, whose elements are easily viewed as bytes: 8-tuples of 0's and 1's. The decoding algorithm depends on the evaluation of a polynomial at the 255 nonzero elements of this field. Roughly speaking, the algorithm produces a polynomial whose degree is the number of errors to be corrected and whose roots are the locations of the errors.

"Finding the roots of a polynomial is part of a long tradition in mathematics," Winograd observes.

With only 255 nonzero elements, an exhaustive search is a reasonable approach—it's like looking for a particular word in a page of text. But there is growing demand to segment data into larger chunks, and that calls for larger fields, such as $GF(2^{12})$. At that point, brute-force search exhausts even the most brutish of computers—keeping up with megabauds of incoming data is simply too expensive.

Linear algebra to the rescue. One of the wonderful accidents of the 0–1 field is that squaring is a linear operation: $(a + b)^2 = a^2 + b^2$, since $2ab = 0$. This means that polynomials in which the exponents are restricted to powers of 2 can be viewed as a matrix applied to a vector: $a_0 + a_1x + a_2x^2 + a_4x^4 + a_8x^8 + \ldots = a_0 + (a_1I + a_2S + a_4S^2 + a_8S^3 + \ldots)x$, where S is the matrix for squaring. Finding the roots of these "linearized" polynomials is readily done with standard linear algebra techniques. All polynomials, it turns out, can be "embedded" in linearized polynomials. The general cubic $x^3 + ax^2 + bx + c$, for example, can be converted into the form $y^3 + py + q$ by substituting y for $x + a$, and then multiplied by y to become $y^4 + py^2 + qy$. The problem is, the embedding tends to introduce lots of superfluous roots. Linear algebra needs a little help.

Idempotents to the rescue. Winograd and Trager have focused on a vector space of "idempotents mod P." If P is the polynomial whose roots are to be found, an idempotent is a (lower-degree) polynomial p such that $p^2 \equiv p \bmod P$. A key theorem is that if p is an idempotent (other than 0 or 1), then the greatest common divisor of p and P—which is relatively easy to compute—is a nontrivial factor of P.

The IBM researchers have worked out efficient algorithms based on idempotents for polynomials up to degree 6. If your work has more than half a dozen errors in it, it may take more than linear algebra to straighten you out.

Zalman Usiskin spoke on another subject in which errors abound: math education. Usiskin, a professor of education at the University of Chicago (he was Lynn Friedman's thesis adviser in the 1980s) and director of the University of Chicago School Mathematics Project, described some of the lessons that have been learned—and some that apparently haven't—from international comparisons of mathematics achievement.

In February, the Third International Mathematics and Science Study (TIMSS) released the results for the 12th-grade level of a test administered in 1994–95 to half a million students from 41 countries in grades 4, 8, and 12. (Not every country participated at all grade levels.) The TIMSS results for similar tests at the eighth- and fourth-grade levels had been announced in 1996 and 1997.

To no one's surprise, the U.S. high school students were near the bottom of the heap. Fourteen countries scored significantly higher on the mathematics exam, according to statistical analysis, four were not significantly different, and only two—Cyprus and South Africa—were significantly below the U.S. 12th graders. For science, the corresponding breakdown was 11 higher, seven similar, and the same two countries below the U.S. However, Usiskin says, "only seven countries met the strict sampling requirements of the international panel, and the U.S. was not one of them, so any comparisons involving the U.S. are suspect."

Things are slightly better at the eighth-grade level: The U.S. sample and many others did meet the international requirements, and U.S. middle schoolers did significantly better at mathematics than their counterparts in seven of 40 countries and "tied" with 13 others. In science, only nine countries did better than the U.S., and 15 did significantly worse. And U.S. fourth graders are a force to be reckoned with. They beat 12 other countries, tied with six, and lagged only seven in mathematics, while in science, only one country—Korea—outperformed the U.S., which tied with five others. (Blow-by-blow analyses can be found at the National Center for Education Statistics Web site, http://nces.ed.gov/timss/.)

Do U.S. kids get dumber as they grow older? Or do the results suggest that the educational reforms of the last decade are beginning to pay off? It's hard to tell. TIMSS was the first study to include fourth graders. SIMS (which had no science component), conducted in 1981, and FIMS, in 1963–64, focused solely on high school students. All that can be said for sure is that the U.S. has a history of mediocre-to-poor performance.

Why? Many favorite explanations have been studied and found wanting, Usiskin says. U.S. students do not spend less time in math and science classes than students in other countries. Nor does large class size seem to be the culprit: Smaller classes are not positively correlated with achievement. U.S. teachers are often said to be less prepared than their counterparts elsewhere. That's also wrong (although they do have higher teaching loads). Could it be the quality of teaching? The SIMS researchers were not sure, and the TIMSS researchers saw it as a problem only at the eighth-grade level.

One possible reason does seem to stand out: curriculum. Schools in the U.S. seem to spend a lot of time rehashing old material. The correlation with subpar performance, reported by the SIMS researchers in the 1980s, has since been verified by the TIMSS and other researchers.

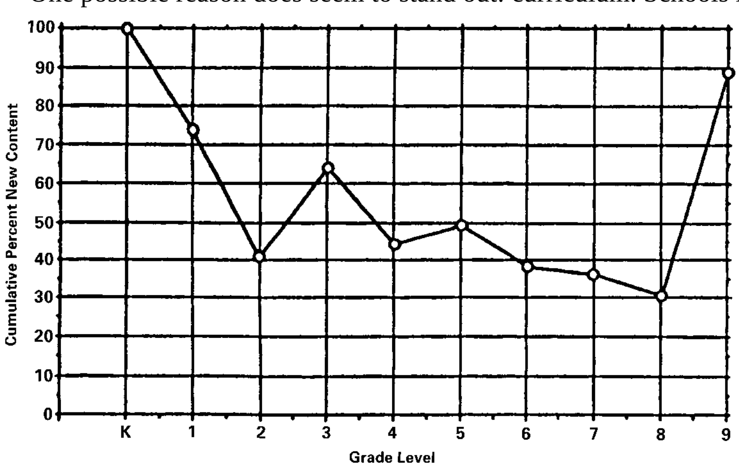

Figure 2. *The "Flanders graph," from James Flanders's 1987 page-by-page study of three mathematics textbook series in wide use in the U.S. Numbers of pages containing any new material decline steadily from kindergarten to eighth grade; the upturn at ninth grade represents Algebra I.*

The rehashing of the U.S. curriculum was demonstrated dramatically in a 1987 study by James Flanders, then a graduate student in mathematics education at the University of Chicago. Flanders took a page-by-page look at the content of three mathematics series widely used in the U.S., and counted the number of pages that contained anything new, not introduced in a previous year. (Textbooks tend to be written so that every page gets the same amount of class time.) He graphed the results from kindergarten through ninth grade (see Figure 2).

The "Flanders graph," as it's come to be known, shows a steady decline—from 100% (by definition) in kindergarten to 30% in eighth grade, with a sharp upturn at ninth grade, where kids run into the buzz-saw of algebra. Given Flanders's generous definition of a page with new content, Usiskin points out, the actual state of affairs is probably far worse.

There are signs of progress, though, Usiskin says. Most significantly, there's been a substantial shift of students in the U.S. into higher-level mathematics classes. In 1978, for example, 37% of students did not go beyond Algebra I; in 1996, the number was down to 20%. The curricular flesh may still be weak, but at least the academic spirit appears more willing.

No work of art is really ever finished. They only stop at good places.

—Henri

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